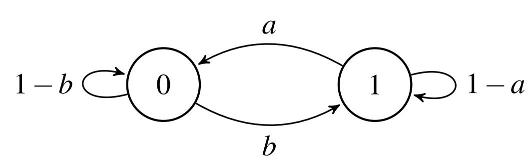
1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

2 Consecutive Flips

Suppose you are flipping a standard coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

(a) Construct an Markov chain that describes the situation with a start state and end state.

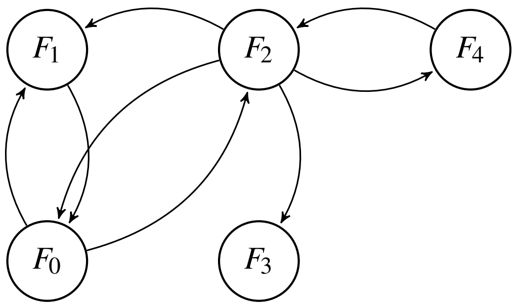
(b) Given that you have flipped a (Tails, Heads) so far, how many expected number of flips?

(c) What is the expected number of flips from the start state?

3 The Dwinelle Labyrinth

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



(a) Will you make it in time if you choose a floor to transition to uniformly at random? (If T_i is the number of steps needed to get to F_3 starting from F_i, where $i \in \{0,1,2,3,4\}$, is $\mathbb{E}[T_0] < 12$?)

(b) Will you make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (If you are considering 1, 2, or 3, you will take each with probabilities $1/7$, $2/7$, $4/7$, respectively.)