1 Aperiodicity

(a) Can you find a finite irreducible aperiodic Markov chain whose distribution does not converge?

(b) Construct a finite Markov chain that is a sequence of i.i.d. random variables. Is it necessarily irreducible and aperiodic? What is its invariant distribution?

2 Allen's Umbrellas

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

We will model this as a Markov chain. Let $\mathscr{X} = \{0,1,2\}$ be the set of states, where the state i represents the number of umbrellas in his current location. Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

3 Faulty Machines

You are trying to use a machine that only works on some days. If on a given day the machine is working, it will break down the next day with probability $0 < b < 1$, and works on the next day $1-b$. If it is not working on a given day, it will work on the next day with probability $0 < r < 1$, and not work on the next day with probability $1-r$. Formulate this process as a Markov chain. As $n \to \infty$, what does the probability that on a given day the machine is working converge to? What properties of the Markov chain allow us to conclude that the probability will actually converge?