CS 70 Spring 2018 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — DIS 9A

1 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$?

(b) What is $\mathbb{P}(X=1)$?

(c) What is $\mathbb{P}(X=2)$?

(d) What is $\mathbb{P}(X=3)$?

(e) Do the answers you computed in parts (a) through (d) add up to 1, as expected?

(f) Compute $\mathbb{E}(X)$ from the definition of expectation.

(g) Suppose we define indicators X_i, $1 \le i \le 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$.

(h) Are the X_i indicators independent? Does this affect your solution to part (g)?

Solution:

(a) We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X=0) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X=0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}$$

(b) We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X=1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

(c) Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X=2)=\frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}}=\frac{72}{5525}$$

(d) Choose 3 queens out of 4.

$$\mathbb{P}(X=3)=\frac{\binom{4}{3}}{\binom{52}{3}}=\frac{1}{5525}$$

(e) We check:

$$\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3)=\frac{4324+1128+72+1}{5525}=1$$

(f) From the definition, $\mathbb{E}(X)=\sum_{k=0}^{3}k\mathbb{P}(X=k)$, so

$$\mathbb{E}(X)=0\cdot\frac{4324}{5525}+1\cdot\frac{1128}{5525}+2\cdot\frac{72}{5525}+3\cdot\frac{1}{5525}=\frac{3}{13}.$$

(g) We know that $\mathbb{E}(X_i)=\mathbb{P}(\text{card } i \text{ is a queen})+0\cdot\mathbb{P}(\text{card } i \text{ is not a queen})=1/13$, so

$$\mathbb{E}(X)=\mathbb{E}(X_1)+\mathbb{E}(X_2)+\mathbb{E}(X_3)=\frac{1}{13}+\frac{1}{13}+\frac{1}{13}=\frac{3}{13}.$$

Notice how much faster it was to compute the expectation using indicators!

(h) No, they are not independent. As an example:

$$\mathbb{P}(X_1=1)\mathbb{P}(X_2=1)=\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$$

However,

$$\mathbb{P}(X_1=1,X_2=1)=\mathbb{P}(\text{the first and second cards are both queens})=\frac{4}{52}\cdot\frac{3}{51}=\frac{1}{221}.$$

Even though the indicators are not independent, this does not change our answer for part (g). Linearity of expectation *always* holds, which makes it an extremely powerful tool.

2 Quadruply-Repeated Ones

We say that a string of bits has k *quadruply-repeated ones* if there are k positions where four consecutive 1's appear in a row. For example, the string 0100111110 has two quadruply-repeated ones.

What is the expected number of quadruply-repeated ones in a random n-bit string, when $n \geq 3$ and all n-bit strings are equally likely?

Solution:

For $i \in \{1,2,\ldots,n-3\}$, let X_i be the indicator random variable that equals 1 if i is a position where four consecutive 1's appear in a row. Observe that

$$\mathbb{E}[X_i] = \mathbb{P}[X_i = 1] = \frac{1}{16} \ .$$

By linearity of expectation the answer is $\sum_{i=1}^{n-3} \mathbb{E}[X_i] = (n-3)/16$.

3 More Aces in a Deck

There are four aces in a deck. Suppose you shuffle the deck; define the random variables:

$$\begin{aligned}
X_1 &= \text{number of non-ace cards before the first ace} \\
X_2 &= \text{number of non-ace cards between the first and second ace} \\
X_3 &= \text{number of non-ace cards between the second and third ace} \\
X_4 &= \text{number of non-ace cards between the third and fourth ace} \\
X_5 &= \text{number of non-ace cards after the fourth ace}
\end{aligned}$$

1. What is $X_1 + X_2 + X_3 + X_4 + X_5$?
2. Argue that the X_i random variables all have the same distribution. Are they independent?
3. Use the results of the previous parts to compute $\mathbb{E}(X_1)$.

Solution:

1. $X_1 + X_2 + X_3 + X_4 + X_5$ must sum up to the number of non-ace cards in the deck, which is 48.
2. This is true by symmetry; the aces split the deck into 5 equal sections, and by symmetry, the expected number of cards in each section is equal. No they are not independent. For example, if $X_1 = 48$ then we know that $X_2, X_3, X_4 = 0$.
3. Since $X_1 + X_2 + X_3 + X_4 + X_5 = 48$, take the expectation of both sides:

$$\sum_{i=1}^{5} \mathbb{E}(X_i) = 48$$

   Since all of the $\mathbb{E}(X_i)$ are the same, it follows that $\mathbb{E}(X_1) = 48/5$.

4 Airport Revisited

(a) Suppose that there are n airports arranged on a circle. There is a plane departing from each airport, and randomly chooses an airport to its left or right and heads towards it. What is the expected number of empty airports after all planes have landed?

(b) Now suppose thast we still have n airports, but instead of sitting on a circle, they form a general graph, where each airport is denoted by a vertex, and an edge between two airports indicates that a flight is permitted between them. There is a plane departing from each airport and randomly chooses a neighboring destination where a flight is permitted. What is the expected number of empty airports after all planes have landed? (Express your answer in terms of $N(i)$ - the set of neighboring airports of airport i, and $\deg(i)$ - the number of neighboring airports of airport i).

Solution:

(a) Let X_i be the indicator variable denoting whether airport i ends up empty. This can happen if and only if planes from both of its neighboring airports are flying elsewhere, and this happens with a probability of $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Hence the expected number of empty airports is

$$\mathbb{E}[X_1 + \ldots + X_n] = \frac{n}{4}$$

(b) Similar to the previous part, we now have $\mathbb{E}[X_i] = P(X_i = 1) = \prod_{j \in N(i)} \left(1 - \frac{1}{\deg(j)}\right)$. Hence

$$\mathbb{E}[X_1 + \ldots + X_n] = \sum_{i=1}^{n} \prod_{j \in N(i)} \left(1 - \frac{1}{\deg(j)}\right)$$