1 Numbered Balls

Suppose you have a bag containing seven balls numbered $0, 1, 1, 2, 3, 5, 8$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record their total. What is the expected value of the total that you record?

Solution:

(a) Let X be the number that you record. Each ball is equally likely to be chosen, so

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x) = 0 \times \frac{1}{7} + 1 \times \frac{2}{7} + 2 \times \frac{1}{7} + 3 \times \frac{1}{7} + 5 \times \frac{1}{7} + 8 \times \frac{1}{7} = \frac{20}{7}.$$

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome ω for which $X(\omega) = 20/7$).

(b) Let X_1 be the number on the first ball that you pull out, and X_2 be the number on the second ball that you pull out. Then $X = X_1 + X_2$, and

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{20}{7} + \frac{20}{7} = \frac{40}{7}$$

where the second equality applies linearity of expectation. Note that using linearity of expectation does *not* require X_1 and X_2 to be independent! Indeed, X_1 and X_2 are not independent because $\mathbb{P}(X_1 = 0) = 1/7$ but $\mathbb{P}(X_1 = 0 \mid X_2 = 0) = 0$.

2 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X = 0)$?

(b) What is $\mathbb{P}(X = 1)$?

(c) What is $\mathbb{P}(X = 2)$?

(d) What is $\mathbb{P}(X = 3)$?

(e) Do the answers you computed in parts (a) through (d) add up to 1, as expected?

(f) Compute $\mathbb{E}(X)$ from the definition of expectation.

(g) Suppose we define indicators X_i, $1 \le i \le 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$ using linearity of expectation.

(h) Are the X_i indicators independent? Does this affect your solution to part (g)?

Solution:

(a) We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X = 0) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X = 0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}$$

(b) We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X = 1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

(c) Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X = 2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}$$

(d) Choose 3 queens out of 4.

$$\mathbb{P}(X = 3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}$$

(e) We check:

$$\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) = \frac{4324 + 1128 + 72 + 1}{5525} = 1$$

(f) From the definition, $\mathbb{E}(X) = \sum_{k=0}^{3} k\mathbb{P}(X = k)$, so

$$\mathbb{E}(X) = 0 \cdot \frac{4324}{5525} + 1 \cdot \frac{1128}{5525} + 2 \cdot \frac{72}{5525} + 3 \cdot \frac{1}{5525} = \frac{3}{13}.$$

(g) We know that $\mathbb{E}(X_i) = \mathbb{P}(\text{card } i \text{ is a queen}) + 0 \cdot \mathbb{P}(\text{card } i \text{ is not a queen}) = 1/13$, so

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = \frac{1}{13} + \frac{1}{13} + \frac{1}{13} = \frac{3}{13}.$$

Notice how much faster it was to compute the expectation using indicators!

(h) No, they are not independent. As an example:

$$\mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

However,

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(\text{the first and second cards are both queens}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}.$$

Even though the indicators are not independent, this does not change our answer for part (g). Linearity of expectation *always* holds, which makes it an extremely powerful tool.

3 More Aces in a Deck

There are four aces in a deck. Suppose you shuffle the deck; define the random variables:

$$\begin{aligned}
X_1 &= \text{number of non-ace cards before the first ace} \\
X_2 &= \text{number of non-ace cards between the first and second ace} \\
X_3 &= \text{number of non-ace cards between the second and third ace} \\
X_4 &= \text{number of non-ace cards between the third and fourth ace} \\
X_5 &= \text{number of non-ace cards after the fourth ace}
\end{aligned}$$

1. What is $X_1 + X_2 + X_3 + X_4 + X_5$?
2. Argue that the X_i random variables all have the same distribution. Are they independent?
3. Use the results of the previous parts to compute $\mathbb{E}(X_1)$.

Solution:

1. $X_1 + X_2 + X_3 + X_4 + X_5$ must sum up to the number of non-ace cards in the deck, which is 48.
2. This is true by symmetry; the aces split the deck into 5 equal sections, and by symmetry, the expected number of cards in each section is equal. No they are not independent. For example, if $X_1 = 48$ then we know that $X_2, X_3, X_4 = 0$.
3. Since $X_1 + X_2 + X_3 + X_4 + X_5 = 48$, take the expectation of both sides:

$$\sum_{i=1}^{5} \mathbb{E}(X_i) = 48$$

Since all of the $\mathbb{E}(X_i)$ are the same, it follows that $\mathbb{E}(X_1) = 48/5$.