1 Box of Marbles

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly, and pick a marble what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

Solution:

(a) Let B be the event that the picked marble is blue, R be the event that it is red, A_1 be the event that the marble is picked from box 1, and A_2 be the event that the marble is picked from box 2. Therefore we want to calculate $\mathbb{P}(B)$. By total probability,

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)P(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) = 0.5 \times 0.1 + 0.5 \times 0.5 = 0.3.$$

(b) In this part, we want to find $\mathbb{P}(A_1 \mid B)$. By Bayes rule,

$$\mathbb{P}(A_1 \mid B) = \frac{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1)}{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2)} = \frac{0.1 \times 0.5}{0.5 \times 0.1 + 0.5 \times 0.5} = \frac{1}{6}.$$

(c) Let B_1 be the event that first marble is blue, R_1 be the event that the first marble is red, and B_2 be the event that second marble is blue without looking at the color of first marble. We want to find $\mathbb{P}(B_2)$. By total probability,

$$\mathbb{P}(B_2) = \mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(B_2 \mid R_1)\mathbb{P}(R_1) = \frac{99}{999} \times 0.1 + \frac{100}{999} \times 0.9 = 0.1.$$

More generally, one can see that the probability that the n-th marble picked from box 1 is blue with probability 0.1. This is clear by symmetry: all the permutations of the 1000 marbles have the same probability, so the probability that the n-th marble is blue is the same as the probability that the first marble is blue.

2 Duelling Meteorologists

Tom is a meteorologist in New York. On days when it snows, Tom correctly predicts the snow 70% of the time. When it doesn't snow, he correctly predicts no snow 95% of the time. In New York, it snows on 10% of all days.

(a) If Tom says that it is going to snow, what is the probability it will actually snow?

(b) Let A be the event that, on a given day, Tom predicts the weather correctly. What is $\mathbb{P}(A)$?

(c) Tom's friend Jerry is a meteorologist in Alaska. Jerry claims that she is a better meteorologist than Tom even though her overall accuracy is lower. After looking at their records, you determine that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny day. Give an instance of the situation described above. *Hint: what is the weather like in Alaska?*

Solution:

(a) Let S be the event that it snows and T be the event that Tom predicts snow.

$$\begin{aligned}\mathbb{P}(S|T) &= \frac{\mathbb{P}(S\cap T)}{\mathbb{P}(T)} \\ &= \frac{\mathbb{P}(S)\cdot\mathbb{P}(T|S)}{\mathbb{P}(S\cap T)+\mathbb{P}(\overline{S}\cap T)} \\ &= \frac{\frac{1}{10}\times\frac{7}{10}}{\frac{1}{10}\times\frac{7}{10}+\frac{9}{10}\times\frac{5}{100}} = \frac{14}{23}\end{aligned}$$

(b)

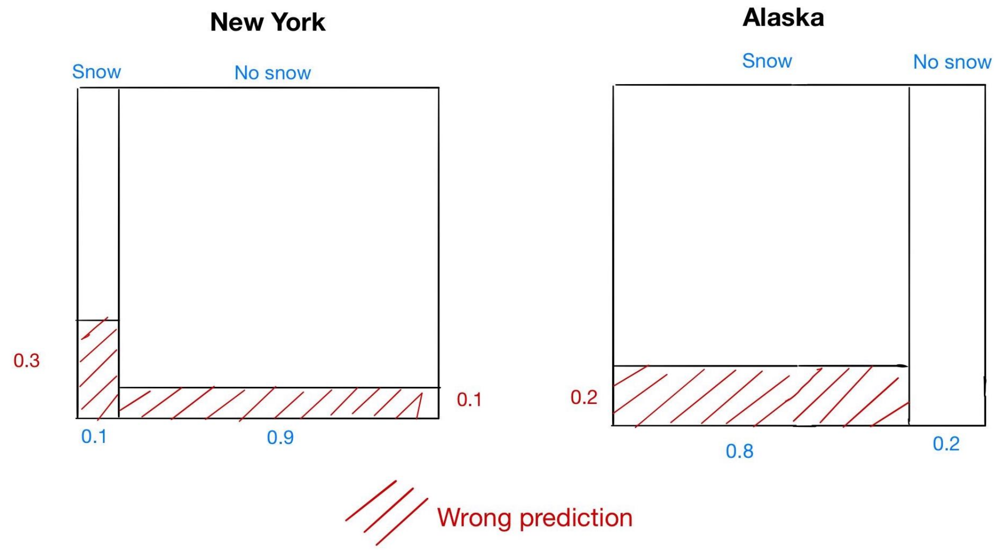
$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(S\cap T)+\mathbb{P}(\overline{S}\cap\overline{T}) \\ &= \frac{1}{10}\times\frac{7}{10}+\frac{9}{10}\times\frac{95}{100} = \frac{37}{40}\end{aligned}$$

(c) Even though Jerry's overall accuracy is lower, it is still possible that she is a better meteorologist if the weather is different.

For example, let's assume that it snows 80% of days in Alaska.

- When it snows, Jerry correctly predicts snow 80% of the time.
- When it doesn't snow, Jerry correctly predicts no snow 100% of the time.

Jerry's overall accuracy turns out to be less than Tom's even though she is better at predicting both categories! The following diagram gives an illustration of the situation. The intuition is that Jerry's error gets penalized more heavily than Tom because it snows more often in Alaska.



For more info on this kind of phenomena, check out Simpson's Paradox!

3 Binary Conditional Probabilities

Let us consider a sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$ of size $N > 2$, and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

If for every subset $A \subset \Omega$ of size $|A| = 2$ and every outcome $\omega \in \Omega$ it is true that $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A)$, then is it necessarily true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)

Solution: Yes, this is indeed true. To see why, let's take the subset $A = \{\omega_i, \omega_j\}$ for some $i, j \in \{1, \ldots, N\}$ and compute: For any $k \in \{1, 2\}$, we have $\mathbb{P}_k(\omega_i \mid A) = \frac{\mathbb{P}_k(\omega_i)}{\mathbb{P}_k(A)}$. Since this expression (by assumption) is the same for $k = 1$ and $k = 2$, we conclude that $\frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_2(\omega_i)} = \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)}$. Repeating the reasoning for ω_j, we similarly find that $\frac{\mathbb{P}_1(\omega_j)}{\mathbb{P}_2(\omega_j)} = \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)}$, and whence $\frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_1(\omega_j)} = \frac{\mathbb{P}_2(\omega_i)}{\mathbb{P}_2(\omega_j)}$. Since this is true for any $i, j \in \{1, \ldots, N\}$, we can sum over i to get

$$\frac{1}{\mathbb{P}_1(\omega_j)} = \sum_{i=1}^{N} \frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_1(\omega_j)} = \sum_{i=1}^{N} \frac{\mathbb{P}_2(\omega_i)}{\mathbb{P}_2(\omega_j)} = \frac{1}{\mathbb{P}_2(\omega_j)},$$

which shows that $\mathbb{P}_1(\omega_j) = \mathbb{P}_2(\omega_j)$ for all $j \in \{1, \ldots, N\}$.