1  DeMorgan’s Laws

Use truth tables to show that \( \neg(A \lor B) \equiv \neg A \land \neg B \) and \( \neg(A \land B) \equiv \neg A \lor \neg B \). These two equivalences are known as DeMorgan’s Laws.

Solution:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A \lor B</th>
<th>\neg(A \lor B)</th>
<th>\neg A \land \neg B</th>
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<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A \land B</th>
<th>\neg(A \land B)</th>
<th>\neg A \lor \neg B</th>
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2  Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification. Recall that \( \mathbb{N} \) refers to the set of natural numbers, \( \mathbb{Q} \) refers to the set of rational numbers, \( \mathbb{Z} \) refers to the set of integers, \( \mathbb{C} \) refers to the set of complex numbers, and \( x \mid y \) denotes that \( x \) divides \( y \).

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) \( (\forall x \in \mathbb{R}) (x \in \mathbb{C}) \)

(e) \( (\forall x \in \mathbb{Z}) (((2 \mid x) \lor (3 \mid x)) \implies (6 \mid x)) \)

(f) \( (\forall x \in \mathbb{N}) ((x > 7) \implies ((\exists a, b \in \mathbb{N}) (a + b = x))) \)

Solution:
(a) \((\exists x \in \mathbb{R}) (x \notin \mathbb{Q})\), or equivalently \((\exists x \in \mathbb{R}) \neg(x \in \mathbb{Q})\). This is true, and we can use \(\pi\) as an example to prove it.

(b) \((\forall x \in \mathbb{Z}) ((x \in \mathbb{N}) \lor (x < 0)) \land \neg((x \in \mathbb{N}) \land (x < 0))\). This is true, since we define the naturals to contain all integers which are not negative.

(c) \((\forall x \in \mathbb{N}) ((6 \mid x) \implies ((2 \mid x) \lor (3 \mid x)))\). This is true, since any number divisible by 6 can be written as \(6k = (2 \cdot 3)k = 2(3k)\), meaning it must also be divisible by 2.

(d) All real numbers are complex numbers. This is true, since any real number \(x\) can equivalently be written as \(x + 0i\).

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take \(a = x\) and \(b = 0\).

(Aside: this is a reference to the very weak Goldback Conjecture (https://xkcd.com/1310/).)

3 Preserving Set Operations

For a function \(f\), define the image of a set \(X\) to be the set \(f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}\). Define the inverse image or preimage of a set \(Y\) to be the set \(f^{-1}(Y) = \{x \mid f(x) \in Y\}\). Prove the following statements, in which \(A\) and \(B\) are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

*Hint: For sets \(X\) and \(Y\), \(X = Y\) if and only if \(X \subseteq Y\) and \(Y \subseteq X\). To prove that \(X \subseteq Y\), it is sufficient to show that \((\forall x) ((x \in X) \implies (x \in Y))\).*

(a) \(f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)\).

(b) \(f(A \cup B) = f(A) \cup f(B)\).

**Solution:**

In order to prove equality \(A = B\), we need to prove that \(A\) is a subset of \(B\), \(A \subseteq B\) and that \(B\) is a subset of \(A\), \(B \subseteq A\). To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose \(x\) is such that \(f(x) \in A \cup B\). Then either \(f(x) \in A\), in which case \(x \in f^{-1}(A)\), or \(f(x) \in B\), in which case \(x \in f^{-1}(B)\), so in either case we have \(x \in f^{-1}(A) \cup f^{-1}(B)\). This proves that \(f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)\).

Now, suppose that \(x \in f^{-1}(A) \cup f^{-1}(B)\). Suppose, without loss of generality, that \(x \in f^{-1}(A)\). Then \(f(x) \in A\), so \(f(x) \in A \cup B\), so \(x \in f^{-1}(A \cup B)\). The argument for \(x \in f^{-1}(B)\) is the same. Hence, \(f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)\).
(b) Suppose that \(x \in A \cup B\). Then either \(x \in A\), in which case \(f(x) \in f(A)\), or \(x \in B\), in which case \(f(x) \in f(B)\). In either case, \(f(x) \in f(A) \cup f(B)\), so \(f(A \cup B) \subseteq f(A) \cup f(B)\).

Now, suppose that \(y \in f(A) \cup f(B)\). Then either \(y \in f(A)\) or \(y \in f(B)\). In the first case, there is an element \(x \in A\) with \(f(x) = y\); in the second case, there is an element \(x \in B\) with \(f(x) = y\). In either case, there is an element \(x \in A \cup B\) with \(f(x) = y\), which means that \(y \in f(A \cup B)\). So \(f(A) \cup f(B) \subseteq f(A \cup B)\).