CS 70 Discrete Mathematics and Probability Theory
Spring 2020 DIS 0B

1 Contraposition

Prove the statement "if $a+b<c+d$, then $a<c$ or $b<d$".

Solution:

The implication we're trying to prove is $(a+b<c+d) \implies ((a<c) \vee (b<d))$, so the contrapositive is $((a \geq c) \wedge (b \geq d)) \implies (a+b \geq c+d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a+b \geq c+d$, which is exactly what we wanted.

2 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n>m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0,1,\ldots,n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0,1,\ldots,n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

3 Prime Form

Prove that every prime number $m>3$ is either of the form $6k+1$ or $6k-1$ for some integer k.

Solution:

First we note that any integer can be written in one of the forms $6k, 6k+1, 6k+2, 6k+3.6k+4$ and $6k+5$. (Note that $6k+5$ is equal to $6(k+1)-1$. Since k is arbitary, we can treat these as equivalent forms).

We can instead prove the contrapositive: that any integer $m > 3$ of the forms $6k, 6k+2, 6k+3, 6k+4$ must be composite. We note that $6k, 6k+2, 6k+4$ can each be written as $2k$ and $6k+3$ is as $3k$ for an appropriate $k > 0$. Thus our original claim is true as well.