1 DeMorgan’s Laws

Use truth tables to show that \( \neg(A \lor B) \equiv \neg A \land \neg B \) and \( \neg(A \land B) \equiv \neg A \lor \neg B \). These two equivalences are known as DeMorgan’s Laws.

**Solution:**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ∨ B</th>
<th>¬(A ∨ B)</th>
<th>¬A ∧ ¬B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>T</td>
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<tr>
<th>A</th>
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2 XOR

The truth table of XOR (denoted by \( \oplus \)) is as follows.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ⊕ B</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
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<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

1. Express XOR using only \((\land, \lor, \neg)\) and parentheses.

2. Does \((A \oplus B)\) imply \((A \lor B)\)? Explain briefly.

3. Does \((A \lor B)\) imply \((A \oplus B)\)? Explain briefly.

**Solution:**

1. These are all correct:
• $A \oplus B = (A \land \neg B) \lor (\neg A \land B)$

Notice that there are only two instances when $A \oplus B$ is true: (1) when $A$ is true and $B$ is false, or (2) when $B$ is true and $A$ is false. The clause $(A \land \neg B)$ is only true when (1) is, and the clause $(\neg A \land B)$ is only true when (2) is.

• $A \oplus B = (A \lor B) \land (\neg A \lor \neg B)$

Another way to think about XOR is that exactly one of $A$ and $B$ needs to be true. This also means exactly one of $\neg A$ and $\neg B$ needs to be true. The clause $(A \lor B)$ tells us at least one of $A$ and $B$ needs to be true. In order to ensure that one of $A$ or $B$ is also false, we need the clause $(\neg A \lor \neg B)$ to be satisfied as well.

• $A \oplus B = (A \lor B) - (A \land B)$

This is the same as the previous, with De Morgan’s law applied to equate $(\neg A \lor \neg B)$ to $(\neg(A \land B))$.

2. Yes. $(A \oplus B) \implies (A \land \neg B) \lor (\neg A \land B) \implies (A \lor B)$.

3. No. When $A$ and $B$ are both true, then $(A \lor B)$ is true, but $(A \oplus B)$ is false.

3 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

(Hint: The Pigeonhole Principle states that if $n$ items are placed in $m$ containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n - 1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly $i$ friends at the party. In particular, there is one person who has $n - 1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to $n$ possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, \ldots, n-1$. The objects assigned to these containers are the people at the party. However, containers $0, n - 1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning $n$ people to at most $n - 1$ containers, and by the pigeonhole principle, at least one of the $n - 1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.
4 Proof Practice

(a) Prove that \( \forall n \in \mathbb{N}, \) if \( n \) is odd, then \( n^2 + 1 \) is even.

(b) Prove that \( \forall x, y \in \mathbb{R}, \) \( \min(x, y) = \frac{(x + y - |x - y|)}{2}. \)

(c) Prove that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \)

(d) Suppose \( A \subseteq B. \) Prove \( \mathcal{P}(A) \subseteq \mathcal{P}(B). \)

Solution:

(a) We will use a direct proof. Assume \( n \) is odd. By the definition of odd numbers, \( n = 2k + 1 \) for some natural number \( k. \) Substituting into the expression \( n^2 + 1, \) we get \( (2k + 1)^2 + 1 = 4k^2 + 4k + 2. \) This can be rewritten as \( 2 \times (2k^2 + 2k + 1). \) Since \( 2k^2 + 2k + 1 \) is a natural number, by the definition of even numbers, \( n^2 + 1 \) is even.

(b) We will use a proof by cases. We know the following about the absolute value function for real number \( z. \)

\[
|z| = \begin{cases} 
  z, & z \geq 0 \\
  -z, & z < 0 
\end{cases}
\]

Case 1: \( x < y. \) This means \( |x - y| = y - x. \) Substituting this into the formula on the right hand side, we get

\[
\frac{x + y - y + x}{2} = x = \min(x, y).
\]

Case 2: \( x \geq y. \) This means \( |x - y| = x - y. \) Substituting this into the formula on the right hand side, we get

\[
\frac{x + y - x + y}{2} = y = \min(x, y).
\]

(c) 

\[
\sum_{i=1}^{n} i = 1 + 2 + \cdots + n
\]

\[
2 \sum_{i=1}^{n} i = (1 + n) + (2 + (n - 1)) + \cdots + (n + 1) = (n + 1)n
\]

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]

(d) Suppose \( A' \in \mathcal{P}(A), \) that is, \( A' \subseteq A \) (by the definition of the power set). We must prove that for any such \( A', \) we also have that \( A' \in \mathcal{P}(B), \) that is, \( A' \subseteq B. \)

Let \( x \in A'. \) Then, since \( A' \subseteq A, x \in A. \) Since \( A \subseteq B, x \in B. \) We have shown \( (\forall x \in A') x \in B, \) so \( A' \subseteq B. \)

Since the previous argument works for any \( A' \subseteq A, \) we have proven \( (\forall A' \in \mathcal{P}(A)) A' \in \mathcal{P}(B). \)
So, we conclude \( \mathcal{P}(A) \subseteq \mathcal{P}(B) \) as desired.