1 Probabilistic Bounds

A random variable $X$ has variance $\text{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of $X$ is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) $\mathbb{E}[X^2] = 13$.
(b) $\mathbb{P}[X = 2] > 0$.
(c) $\mathbb{P}[X \geq 2] = \mathbb{P}[X \leq 2]$.
(d) $\mathbb{P}[X \leq 1] \leq 8/9$.
(e) $\mathbb{P}[X \geq 6] \leq 9/16$.

Solution:

(a) TRUE. Since $9 = \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 2^2$, we have $\mathbb{E}[X^2] = 9 + 4 = 13$.

(b) FALSE. It is not necessary for a random variable to be able to take on its mean as a value. Construct a random variable $X$ that satisfies the conditions in the question but does not take on the value 2. A simple example would be a random variable that takes on 2 values, where $\mathbb{P}[X = a] = \mathbb{P}[X = b] = 1/2$, and $a \neq b$. The expectation must be 2, so we have $a/2 + b/2 = 2$. The variance is 9, so $\mathbb{E}[X^2] = 13$ (from Part (a)) and $a^2/2 + b^2/2 = 13$. Solving for $a$ and $b$, we get $\mathbb{P}[X = -1] = \mathbb{P}[X = 5] = 1/2$ as a counterexample.

(c) FALSE. The median of a random variable is not necessarily the mean, unless it is symmetric. Construct a random variable $X$ that satisfies the conditions in the question but does not have an equal chance of being less than or greater than 2. A simple example would be a random variable that takes on 2 values, where $\mathbb{P}[X = a] = p, \mathbb{P}[X = b] = 1 - p$. Here, we use the same approach as part (b) except with a generic $p$, since we want $p \neq 1/2$. The expectation must be 2, so we have $pa + (1 - p)b = 2$. The variance is 9, so $\mathbb{E}[X^2] = 13$ and $pa^2 + (1 - p)b^2 = 13$. Solving for $a$ and $b$, we find the relation $b = 2 \pm 3/\sqrt{x}$, where $x = (1 - p)/p$. Then, we can find an example by plugging in values for $x$ so that $a, b \leq 10$ and $p \neq 1/2$. One such counterexample is $\mathbb{P}[X = -7] = 1/10, \mathbb{P}[X = 3] = 9/10$.

(d) TRUE. Let $Y = 10 - X$. Since $X$ is never exceeds 10, $Y$ is a non-negative random variable. By Markov’s inequality,

$$\mathbb{P}[10 - X \geq a] = \mathbb{P}[Y \geq a] \leq \frac{\mathbb{E}[Y]}{a} = \frac{\mathbb{E}[10 - X]}{a} = \frac{8}{a}.$$
Setting \( a = 9 \), we get \( P[X \leq 1] = P[10 - X \geq 9] \leq 8/9. \)

(e) TRUE. Chebyshev’s inequality says \( P[|X - \mathbb{E}[X]| \geq a] \leq \text{Var}(X)/a^2 \). If we set \( a = 4 \), we have

\[
P[|X - 2| \geq 4] \leq \frac{9}{16}.
\]

Now we observe that \( P[X \geq 6] \leq P[|X - 2| \geq 4] \), because the event \( X \geq 6 \) is a subset of the event \( |X - 2| \geq 4 \).

2 Inequality Practice

(a) \( X \) is a random variable such that \( X > -5 \) and \( \mathbb{E}[X] = -3 \). Find an upper bound for the probability of \( X \) being greater than or equal to \(-1\).

(b) You roll a die 100 times. Let \( Y \) be the sum of the numbers that appear on the die throughout the 100 rolls. Compute \( \text{Var}(Y) \). Then use Chebyshev’s inequality to bound the probability of the sum \( Y \) being greater than 400 or less than 300.

Solution:

(a) We want to use Markov’s Inequality, but recall that Markov’s Inequality only works with non-negative random variables. So, we define a new random variable \( Y = X + 5 \), where \( Y \) is always non-negative, so we can use Markov’s on \( Y \). By linearity of expectation, \( \mathbb{E}[Y] = -3 + 5 = 2 \). So, \( P[Y \geq 4] \leq 2/4 = 1/2 \).

(b) Let \( Y_i \) be the number on the die for the \( i \)th roll, for \( i = 1, \ldots, 100 \). Then, \( Y = \sum_{i=1}^{100} Y_i \). By linearity of expectation, \( \mathbb{E}[Y] = \sum_{i=1}^{100} \mathbb{E}[Y_i] \).

\[
\mathbb{E}[Y_i] = \sum_{j=1}^{6} j \cdot \mathbb{P}[Y_i = j] = \sum_{j=1}^{6} j \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{j=1}^{6} j = \frac{1}{6} \cdot 21 = \frac{7}{2}
\]

Then, we have \( \mathbb{E}[Y] = 100 \cdot (7/2) = 350. \)

\[
\mathbb{E}[Y_i^2] = \sum_{j=1}^{6} j^2 \cdot \mathbb{P}[Y_i = j] = \sum_{j=1}^{6} j^2 \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{j=1}^{6} j^2 = \frac{1}{6} \cdot 91 = \frac{91}{6}
\]

Then, we have

\[
\text{Var}(Y_i) = \mathbb{E}[Y_i^2] - \mathbb{E}[Y_i]^2 = \frac{91}{6} - \left( \frac{7}{2} \right)^2 = \frac{35}{12}.
\]

Since the \( Y_i \)s are independent, and therefore uncorrelated, we can add the \( \text{Var}(Y_i) \)s to get \( \text{Var}(Y) = 100(35/12) \).

Putting it all together, we use Chebyshev’s to get

\[
P[|X - 350| \geq 50] \leq \frac{100(35/12)}{50^2} = \frac{7}{60}.
\]
3 Tightness of Inequalities

(a) Show by example that Markov’s inequality is tight; that is, show that given \( k > 0 \), there exists a discrete non-negative random variable \( X \) such that \( \mathbb{P}(X \geq k) = \frac{\mathbb{E}[X]}{k} \).

(b) Show by example that Chebyshev’s inequality is tight; that is, show that given \( k \geq 1 \), there exists a random variable \( X \) such that \( \mathbb{P}(|X - \mathbb{E}[X]| \geq k \sigma) = \frac{1}{k^2} \), where \( \sigma^2 = \text{Var} X \).

Solution:

(a) In the proof of Markov’s Inequality (\( \mathbb{P}[X \geq \alpha] \leq \frac{\mathbb{E}[X]}{\alpha} \)), the first time we lose equality is at this step:

\[
\mathbb{E}[X] = \sum_a (a \cdot \mathbb{P}[X = a]) \geq \sum_{a \geq \alpha} (a \cdot \mathbb{P}[X = a])
\]

We get an inequality because we drop all \( a \cdot \mathbb{P}[X = a] \) terms where \( a < \alpha \). Thus, we can only maintain equality if all of these dropped terms were actually 0. This would mean either \( a = 0 \) or \( \mathbb{P}[X = a] = 0 \) for an \( a > 0 \), which means \( X \) can put probability on 0, but should put no probability on any other value < \( \alpha \).

The next time we lose equality in the proof is the step following the one above:

\[
\sum_{a \geq \alpha} (a \cdot \mathbb{P}[X = a]) \geq \alpha \cdot \sum_{a \geq \alpha} \mathbb{P}[X = a]
\]

We get an inequality because we treat all \( a \geq \alpha \) in the summation as just \( \alpha \), so we can pull out the \( \alpha \) term. The only way for us to maintain equality is if we never have to substitute \( \alpha \) for some larger \( a \). This tells us that \( X \) should not put probability on any value > \( \alpha \).

Both of these facts drive the intuition behind our example: that \( X \) can only take values 0 and \( \alpha \).

Let \( X \) be the random variable which is 0 with probability \( 1 - p \) and \( k \) with probability \( p \), where \( k > 0 \). Then, \( \mathbb{E}[X] = kp \), and Markov’s inequality says

\[
\mathbb{P}(X \geq k) \leq \frac{\mathbb{E}[X]}{k} = \frac{kp}{k} = p,
\]

which is tight.

(b) The proof of Chebyshev’s Inequality (\( \mathbb{P}[|X - \mathbb{E}[X]| \geq \alpha] \leq \frac{\text{Var}(X)}{\alpha^2} \)) comes from an application of Markov’s Inequality to the variable \( Y = (X - \mathbb{E}[X])^2 \) being \( \geq \alpha^2 \). The only ways we can lose equality in the proof of Chebyshev’s is if we lose equality in the application of Markov! Therefore, we need the variable \( Y \) to satisfy the conditions from Part (a) that ensure the application of Markov will be tight. To recap, we would need \( Y \) to only take values 0 and \( \alpha^2 \). Thus, \( (X - \mathbb{E}[X]) \) can take on the values \( \{-\alpha, 0, \alpha\} \).

Let

\[
X = \begin{cases}
-a & \text{with probability } k^{-2}/2 \\
 a & \text{with probability } k^{-2}/2 \\
 0 & \text{with probability } 1 - k^{-2}
\end{cases}
\]
for \( a > 0 \). Note that \( \text{Var} X = a^2 k^{-2} \), so \( k \sigma = a \), so Chebyshev’s inequality gives

\[
P(|X - \mathbb{E}[X]| \geq k \sigma) = P(|X - \mathbb{E}[X]| \geq a) \leq \frac{1}{k^2},
\]

which is tight.