1 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game $A$ 10 times and game $B$ 20 times. Each time you play game $A$, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game $B$ is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence “book” appears?

2 Joint Distributions

(a) Give an example of discrete random variables $X$ and $Y$ with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of $X$ and $Y$.

(b) Give an example of discrete random variables $X$ and $Y$ that (i) are not independent and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of $X$ and $Y$. 
3 Ball in Bins

You are throwing \( k \) balls into \( n \) bins. Let \( X_i \) be the number of balls thrown into bin \( i \).

(a) What is \( \mathbb{E}[X_i] \)?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when two balls land in the same bin (if there are \( n \) balls in a bin, count that as \( n - 1 \) collisions). What is the expected number of collisions?