1 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX+b$, where a and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu+b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Solution:

Problem and solution taken from *A First Course in Probability* by Sheldon Ross, 8th edition.

Let $a > 0$.

We start with the cumulative distribution function (CDF) of Y, F_Y.

$$\begin{aligned}
F_Y(x) &= \mathbb{P}[Y \le x] && \text{By definition of CDF} \\
&= \mathbb{P}[aX+b \le x] && \text{Plug in } Y = aX+b \\
&= \mathbb{P}\left[X \le \frac{x-b}{a}\right] && \text{Because } a > 0 \\
&= F_X\left(\frac{x-b}{a}\right) && \text{By definition of CDF. } F_X \text{ denotes the CDF of } X.
\end{aligned} \tag{1}$$

Let f_Y denote the probability density function (PDF) of Y.

$$\begin{aligned}
f_Y(x) &= \frac{\mathrm{d}}{\mathrm{d}x}F_Y(x) && \text{The PDF is the derivative of the CDF.} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}F_X\left(\frac{x-b}{a}\right) && \text{Plug in the result from (1)} \\
&= \frac{1}{a}\cdot f_X\left(\frac{x-b}{a}\right) && \begin{array}{l}\text{PDF is the derivative of CDF.} \\ \text{Apply chain rule, } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-b}{a}\right) = \frac{1}{a}.\end{array} \\
&= \frac{1}{a}\cdot\frac{1}{\sigma\sqrt{2\pi}}\cdot \mathrm{e}^{-((x-b)/a-\mu)^2/(2\sigma^2)} && X \sim \mathcal{N}(\mu, \sigma^2). \\
&= \frac{1}{a\sigma\sqrt{2\pi}}\cdot \mathrm{e}^{-(x-b-a\mu)^2/(2\sigma^2a^2)} && \frac{x-b}{a} - \mu = \frac{1}{a}(x-b-a\mu)
\end{aligned} \tag{2}$$

We have shown that f_Y equals the probability density function of a normal random variable with mean $b+a\mu$ and variance σ^2a^2. So, Y is normally distributed with mean $b+a\mu$ and variance σ^2a^2. The proof is done for $a > 0$. The proof for $a < 0$ is similar.

2 Tails of the Gaussian

In the question we will bounds on the probability that $|X| \geq t$ where X is a Gaussian random variable with mean 0 and variance 1.

(a) Use Chebyshev's inequality to bound the probability that $|X| \geq t$ for $t \geq 1$.

(b) We will now show that it is possible to get much better bounds. Firstly, prove that:

$$\mathbb{P}[|X| \geq t] = 2\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

(c) Now, noting that when $t \geq 1$, we have:

$$\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \leq \int_t^\infty \frac{t}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

Prove a new upper bound on the probability in part (a).

(d) Why is this a better upper bound than the one in part (a)?

Solution:

(a) Applying Chebyshev's inequality, we see that:

$$\mathbb{P}[|X| \geq t] \leq \frac{Var(X)}{t^2} = \frac{1}{t^2}$$

(b) We see that:

$$\begin{aligned}\mathbb{P}[|X| \geq t] &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-t} \exp(-\frac{x^2}{2}) dx + \int_t^\infty \exp(-\frac{x^2}{2}) dx\right) \\ &= 2\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx\end{aligned}$$

where the last line follows from the symmetry of the Gaussian distribution about the origin.

(c) We get the bound on the probability via the following inequality:

$$\mathbb{P}[|X| \geq t] = 2\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \leq 2\int_t^\infty \frac{t}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \leq \exp\left(-\frac{t^2}{2}\right)$$

(d) Comparing the two bounds, we see that when t gets large, the second bound is much better than the first. For example, when $t = \sqrt{2\log n}$, we see that the ratio of the two bounds is:

$$\frac{n}{2\log n}$$

which goes to ∞ as n (and thus t) gets large.

3 Hypothesis testing

We would like to test the hypothesis claiming that a coin is fair, i.e. $P(H) = P(T) = 0.5$. To do this, we flip the coin $n = 100$ times. Let Y be the number of heads in $n = 100$ flips of the coin. We decide to reject the hypothesis if we observe that the number of heads is less than $50 - c$ or larger than $50 + c$. However, we would like to avoid rejecting the hypothesis if it is true; we want to keep the probability of doing so less than 0.05. Please determine c. (*Hints: use the central limit theorem to estimate the probability of rejecting the hypothesis given it is actually true.*)

Solution:

Let X_i be the random variable denoting the result of the i-th flip:

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th flip is head,} \\ 0 & \text{if the } i\text{-th flip is tail.} \end{cases}$$

Then we have $Y = \sum_{i=1}^{n} X_i$. If the hypothesis is true, then $\mu = \mathbb{E}[X_i] = \frac{1}{2}$ and $\sigma^2 = \text{var}(X_i) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. By central limit theorem, we know that

$$P\left(\frac{Y - n\mu}{\sqrt{n\sigma^2}} \leq z\right) \approx \Phi(z)$$

$$P\left(\frac{Y - 100 \cdot \frac{1}{2}}{\sqrt{100 \cdot \frac{1}{4}}} \leq z\right) \approx \Phi(z)$$

$$P\left(\frac{Y - 50}{5} \leq z\right) \approx \Phi(z)$$

where

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x.$$

We will reject the hypothesis when $|Y - 50| > c$. We also want $P(|Y - 50| > c) < 0.05$, or equivalently $P(|Y - 50| \leq c) > 0.95$. We have

$$P(|Y - 50| \leq c) = P\left(\frac{|Y - 50|}{5} \leq \frac{c}{5}\right) \approx 2\Phi(\frac{c}{5}) - 1.$$

The reason this is $\approx 2\Phi(\frac{c}{5}) - 1$ is because the probability we are looking for is the probability that Y is within $\frac{c}{5}$ standard deviations of the mean. By an area argument, we can see that this is $\Phi(\frac{c}{5}) - (1 - \Phi(\frac{c}{5})) = 2\Phi(\frac{c}{5}) - 1$. Let $2\Phi(\frac{c}{5}) - 1 = 0.95$, so $\Phi(\frac{c}{5}) = 0.975$ or $\frac{c}{5} = 1.96$. That is $c = 9.8$ flips. So we see that if we observe more that $50 + 10 = 60$ or less than $50 - 10 = 40$ heads, we can reject the hypothesis.

NORMAL CUMULATIVE DISTRIBUTION FUNCTION

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000