1 Allen's Umbrellas

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

We will model this as a Markov chain. Let $\mathscr{X} = \{0,1,2\}$ be the set of states, where the state i represents the number of umbrellas in his current location. Write down the transition matrix, determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

2 Predicament

Three men are on a boat with cigarettes, but they have no lighter. What do they do?

3 Guess the Polynomial

Remember the mantra "$d+1$ points uniquely determine a degree $\leq d$ polynomial ($d \in \mathbb{N}$)"?

Write down a polynomial (of any degree) with non-negative, integer coefficients. Your TA will then guess the exact polynomial you have written down, using only information about the polynomial at two points!

4 Which Envelope?

You have two envelopes in front of you containing cash. You know that one envelope contains twice as much money as the other envelope (the amount of money in an envelope is an integer). You are allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?

5 Coloring a Sphere

Consider a sphere in which exactly $1/10$ of the surface of the sphere is colored red (the rest of the sphere is blue). Prove that no matter how the blue is distributed upon the sphere, it is always possible to inscribe a cube inside the sphere so that all of its corners are blue.