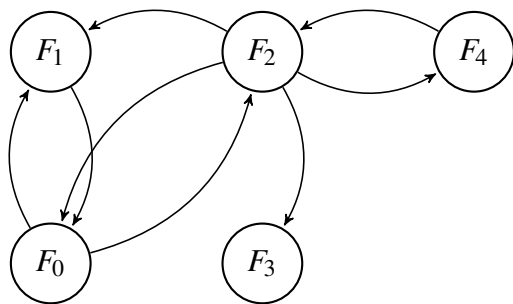


DIS 14B

1 The Dwinelle Labyrinth

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



- (a) Will you make it in time if you choose a floor to transition to uniformly at random? (If T_i is the number of steps needed to get to F_3 starting from F_i , where $i \in \{0, 1, 2, 3, 4\}$, is $\mathbb{E}[T_0] < 12$?)
- (b) Will you make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (If you are considering 1, 2, or 3, you will take each with probabilities $1/7, 2/7, 4/7$, respectively.)

Solution:

- (a) Write out all of the first-step equations.

$$\mathbb{E}[T_0] = 1 + \frac{1}{2} \mathbb{E}[T_1] + \frac{1}{2} \mathbb{E}[T_2]$$

$$\mathbb{E}[T_1] = 1 + \mathbb{E}[T_0]$$

$$\mathbb{E}[T_2] = 1 + \frac{1}{4} \mathbb{E}[T_0] + \frac{1}{4} \mathbb{E}[T_1] + \frac{1}{4} \mathbb{E}[T_3] + \frac{1}{4} \mathbb{E}[T_4]$$

$$\mathbb{E}[T_3] = 0$$

$$\mathbb{E}[T_4] = 1 + \mathbb{E}[T_2]$$

Let us rewrite these equations, before placing it in matrix form.

$$\begin{aligned}
 -1 &= -\mathbb{E}[T_0] + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
 -1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\
 -1 &= -\mathbb{E}[T_2] + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
 0 &= \mathbb{E}[T_3] \\
 -1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2]
 \end{aligned}$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/4 & 1/4 & -1 & 1/4 & 1/4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We can now reduce the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 15$, meaning it will take 15 minutes for us to get to floor 3. Unfortunately, we only have 12 minutes.

(b) Write out all of the first-step equations.

$$\begin{aligned}
 \mathbb{E}[T_0] &= 1 + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\
 \mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\
 \mathbb{E}[T_2] &= 1 + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\
 \mathbb{E}[T_3] &= 0 \\
 \mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2]
 \end{aligned}$$

Let us rewrite these equations, before placing it in matrix form.

$$\begin{aligned}
 -1 &= -\mathbb{E}[T_0] + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\
 -1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\
 -1 &= -\mathbb{E}[T_2] + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\
 0 &= \mathbb{E}[T_3] \\
 -1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2]
 \end{aligned}$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/3 & 2/3 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/15 & 2/15 & -1 & 4/15 & 8/15 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We row reduce to get the following.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 9.75 \\ 0 & 1 & 0 & 0 & 0 & 10.75 \\ 0 & 0 & 1 & 0 & 0 & 7.75 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8.75 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 9.75$, meaning it will take 9.75 minutes for us to get to floor 3. That's fewer than 12 minutes, so if you finished this computation in less than 2 minutes and 15 seconds, you could make it!

2 Guess the Polynomial

Remember the mantra “ $d + 1$ points uniquely determine a degree $\leq d$ polynomial ($d \in \mathbb{N}$)”?

Write down a polynomial (of any degree) with non-negative, integer coefficients. Your TA will then guess the exact polynomial you have written down, using only information about the polynomial at two points!

Solution:

First, ask for the value of the polynomial at 1. Let x be the value of the polynomial evaluated at 1. Next, ask the student for the value of the polynomial at $x + 1$. Expand the student's answer in base $x + 1$, and you will have all of the coefficients of the polynomial.

Explanation: If the polynomial is $P(x) = a_n x^n + \dots + a_0$, where $n = \deg P$, then the base $x + 1$ expansion of $P(x + 1)$ is a number $d_n \dots d_0$, such that

$$P(x + 1) = \sum_{i=0}^n d_i (x + 1)^i.$$

We can read off the coefficients of the polynomials from the base $x + 1$ expansion: the digit d_i in the expansion equals a_i .

The reason why this works is because we know that $x + 1$ is larger than any of the coefficients of the polynomial. Since $x = P(1) = a_n + \dots + a_0$ and all of the coefficients are non-negative, then $x + 1$ is strictly greater than any of the coefficients. Essentially, the purpose of the first step is to gauge how large the coefficients are, and the second step will work as long as you ask for any point which is greater than any individual coefficient (for example, you could just as well ask for the value at $x + 2$, but larger numbers means more difficult arithmetic).

This does not contradict the mantra “ $d + 1$ points uniquely determine a degree $\leq d$ polynomial”. In this case we have restricted the coefficients of our polynomials to a very special set (the non-negative integers), so we can do better than asking for $d + 1$ points.

3 Which Envelope?

You have two envelopes in front of you containing cash. You know that one envelope contains twice as much money as the other envelope (the amount of money in an envelope is an integer). You are allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?

Solution:

Surprisingly, the answer is yes.

First, we are going to choose a random positive integer z . To accomplish this, we flip coins! Let n be the number of flips you need before you see heads. Then, set $z = 2^{n-1} + 0.5$ (okay we lied; this is not an integer, but the 0.5 is just there to break ties later on).

Here is the strategy: look at how much cash is in the envelope you pick. If the amount of cash in the envelope exceeds z , then keep the envelope; otherwise, switch to the other envelope.

Suppose z is smaller than the amount of money in either envelope. Then, you will always keep your original envelope, but since the envelope you chose in the first place is equally likely to be the more lucrative envelope, you still have a $1/2$ probability of choosing the right envelope.

A similar analysis holds if z is greater than the amount of money in either envelope; here you will always switch envelopes, and you have a $1/2$ probability of choosing the right envelope.

However, what happens if z happens to land in between the values of the envelopes? If you initially chose the envelope with less money, then you will switch to the better envelope; if you initially chose the envelope with more money, you will stick with the better envelope. So, in this case, you are guaranteed to end up with the better envelope!

Since z has a positive probability of landing between the values of the envelopes, this strategy gives you a probability of choosing the better envelope which is strictly better than $1/2$.