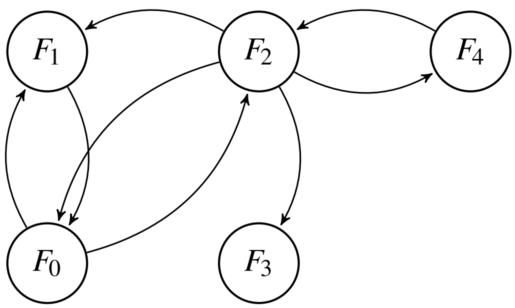
CS 70	Discrete Mathematics and Probability Theory	
Fall 2018	Course Notes	DIS 14B

1 The Dwinelle Labyrinth

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



(a) Will you make it in time if you choose a floor to transition to uniformly at random? (If T_i is the number of steps needed to get to F_3 starting from F_i, where $i \in \{0,1,2,3,4\}$, is $\mathbb{E}[T_0] < 12$?)

(b) Will you make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (If you are considering 1, 2, or 3, you will take each with probabilities $1/7$, $2/7$, $4/7$, respectively.)

Solution:

(a) Write out all of the first-step equations.

$$
\begin{aligned}
\mathbb{E}[T_0] &= 1 + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
\mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\
\mathbb{E}[T_2] &= 1 + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
\mathbb{E}[T_3] &= 0 \\
\mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2]
\end{aligned}
$$

Let us rewrite these equations, before placing it in matrix form.

$$
\begin{aligned}
-1 &= -\mathbb{E}[T_0] + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
-1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\
-1 &= -\mathbb{E}[T_2] + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
0 &= \mathbb{E}[T_3] \\
-1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2]
\end{aligned}
$$

We can rewrite this in matrix form.

$$
P = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/4 & 1/4 & -1 & 1/4 & 1/4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}
$$

We can now reduce the matrix.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{bmatrix}
$$

We see that $\mathbb{E}[T_0] = 15$, meaning it will take 15 minutes for us to get to floor 3. Unfortunately, we only have 12 minutes.

(b) Write out all of the first-step equations.

$$
\begin{aligned}
\mathbb{E}[T_0] &= 1 + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\
\mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\
\mathbb{E}[T_2] &= 1 + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\
\mathbb{E}[T_3] &= 0 \\
\mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2]
\end{aligned}
$$

Let us rewrite these equations, before placing it in matrix form.

$$
\begin{aligned}
-1 &= -\mathbb{E}[T_0] + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\
-1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\
-1 &= -\mathbb{E}[T_2] + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\
0 &= \mathbb{E}[T_3] \\
-1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2]
\end{aligned}
$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/3 & 2/3 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/15 & 2/15 & -1 & 4/15 & 8/15 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We row reduce to get the following.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 9.75 \\ 0 & 1 & 0 & 0 & 0 & 10.75 \\ 0 & 0 & 1 & 0 & 0 & 7.75 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8.75 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 9.75$, meaning it will take 9.75 minutes for us to get to floor 3. That's fewer than 12 minutes, so if you finished this computation in less than 2 minutes and 15 seconds, you could make it!

2 Reflecting Random Walk

Alice starts at vertex 0 and wishes to get to vertex n. When she is at vertex 0 she has a probability of 1 of transitioning to vertex 1. For any other vertex i, there is a probability of $1/2$ of transitioning to $i+1$ and a probability of $1/2$ of transitioning to $i-1$.

(a) What is the expected number of steps Alice takes to reach vertex n? Write down the hitting-time equations, but do not solve them yet.

(b) Solve the hitting-time equations. [*Hint*: Let R_i denote the expected number of steps to reach vertex n starting from vertex i. As a suggestion, try writing R_0 in terms of R_1; then, use this to express R_1 in terms of R_2; and then use this to express R_2 in terms of R_3, and so on. See if you can notice a pattern.]

Solution:

Formulate hitting time equations; the hard part is solving them. R_i represents the expected number of steps to get to vertex n starting from vertex i. In particular, $R_n = 0$ and we are interested in calculating R_0. We have the equations:

$$\begin{aligned}
R_0 &= 1 + R_1, \\
R_1 &= 1 + \frac{1}{2}R_0 + \frac{1}{2}R_2, \\
&\vdots \\
R_i &= 1 + \frac{1}{2}R_{i-1} + \frac{1}{2}R_{i+1}, \\
&\vdots \\
R_{n-1} &= 1 + \frac{1}{2}R_{n-2} + \frac{1}{2}R_n.
\end{aligned}$$

Plug in $R_0 = 1 + R_1$ to the second equation: $R_1 = 1 + 1/2 + (1/2)R_1 + (1/2)R_2$ which then implies $R_1 = 3 + R_2$. In fact, if $R_i = k + R_{i+1}$, then

$$R_{i+1} = 1 + \frac{1}{2}R_i + \frac{1}{2}R_{i+2} = 1 + \frac{k}{2} + \frac{1}{2}R_{i+1} + \frac{1}{2}R_{i+2},$$

which, after moving $(1/2)R_{i+1}$ to the left and multiplying by two, implies $R_{i+1} = k + 2 + R_{i+2}$.

Therefore, $R_0 = 1 + R_1 = 1 + 3 + R_2 = 1 + 3 + 5 + R_3 = \cdots = 1 + 3 + \cdots + 2n - 1 + R_n$ and since $R_n = 0$, we have $R_0 = n^2$.

3 Predicament

Three men are on a boat with cigarettes, but they have no lighter. What do they do?

Solution:

One man throws his cigarette off the boat. The boat is now a cigarette lighter!

4 Which Envelope?

You have two envelopes in front of you containing cash. You know that one envelope contains twice as much money as the other envelope (the amount of money in an envelope is an integer). You are allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?

Solution:

Surprisingly, the answer is yes.

First, we are going to choose a random positive integer z. To accomplish this, we flip coins! Let n be the number of flips you need before you see heads. Then, set $z = 2^{n-1} + 0.5$ (okay we lied; this is not an integer, but the 0.5 is just there to break ties later on).

Here is the strategy: look at how much cash is in the envelope you pick. If the amount of cash in the envelope exceeds z, then keep the envelope; otherwise, switch to the other envelope.

Suppose z is smaller than the amount of money in either envelope. Then, you will always keep your original envelope, but since the envelope you chose in the first place is equally likely to be the more lucrative envelope, you still have a $1/2$ probability of choosing the right envelope.

A similar analysis holds if z is greater than the amount of money in either envlope; here you will always switch envelopes, and you have a $1/2$ probability of choosing the right envelope.

However, what happens if z happens to land in between the values of the envelopes? If you initially chose the envelope with less money, then you will switch to the better envelope; if you initially chose the envlope with more money, you will stick with the better envelope. So, in this case, you are guaranteed to end up with the better envelope!

Since z has a positive probability of landing between the values of the envelopes, this strategy gives you a probability of choosing the better envelope which is strictly better than $1/2$.