1 Stable Matching

Consider the set of jobs $J = \{1, 2, 3\}$ and the set of candidates $C = \{A, B, C\}$ with the following preferences.

<table>
<thead>
<tr>
<th>Jobs</th>
<th>Candidates</th>
<th>Candidates</th>
<th>Jobs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A &gt; B &gt; C</td>
<td>A</td>
<td>2 &gt; 1 &gt; 3</td>
</tr>
<tr>
<td>2</td>
<td>B &gt; A &gt; C</td>
<td>B</td>
<td>1 &gt; 3 &gt; 2</td>
</tr>
<tr>
<td>3</td>
<td>A &gt; B &gt; C</td>
<td>C</td>
<td>1 &gt; 2 &gt; 3</td>
</tr>
</tbody>
</table>

Run the traditional propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work.)

2 Propose-and-Reject Proofs

Prove the following statements about the traditional propose-and-reject algorithm.

(a) In any execution of the algorithm, if a candidate receives a proposal on day $i$, then she receives some proposal on every day thereafter until termination.

(b) In any execution of the algorithm, if a candidate receives no proposal on day $i$, then she receives no proposal on any previous day $j$, $1 \leq j < i$. 
(c) In any execution of the algorithm, there is at least one candidate who only receives a single proposal. (Hint: use the parts above!)

3 Be a Judge

For each of the following statements about the traditional stable matching algorithm with jobs proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

(a) There is a set of preferences for $n$ jobs and $n$ candidates for $n > 1$, such that in a stable matching algorithm execution every job ends up with its least preferred candidate.

(b) In a stable matching instance, if job $J$ and candidate $C$ each put each other at the top of their respective preference lists, then $J$ must be paired with $C$ in every stable pairing.

(c) In a stable matching instance with at least two jobs and two candidates, if job $J$ and candidate $C$ each put each other at the bottom of their respective preference lists, then $J$ cannot be paired with $C$ in any stable pairing.

(d) For every $n > 1$, there is a stable matching instance for $n$ jobs and $n$ candidates which has an unstable pairing in which every unmatched job-candidate pair is a rogue couple.

4 Universal Preference

Suppose that preferences in a stable matching instance are universal: all $n$ jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What pairing do we get from running the algorithm with jobs proposing? Can you prove this happens for all $n$?
(b) What pairing do we get from running the algorithm with candidates proposing?

(c) What does this tell us about the number of stable pairings?