CS 70 Discrete Mathematics and Probability Theory
Spring 2020

Quiz 2

1. [True or False?] For each of the questions, answer TRUE or FALSE. [No need to justify.]

(a) ☐ Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. Thus, $P(1)$ has to be false.

(b) ☐ Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. Then, we cannot say anything about P(50).

(c) ☐ In a stable matching instance where there is a job at the bottom of each candidate's preference list, the job is paired with its least favorite candidate in every stable pairing.

(d) ☐ In a stable matching instance where there is a job at the top of each candidate's preference list, the job is paired with its favorite candidate in every stable pairing.

Solution:

(a) TRUE. Suppose $P(1)$ is true. That provides the base case of the induction, and together with the inductive step this implies that $P(n)$ is true for all n, including for $n = 29$. But this is a contradiction, as $P(29)$ is false, so $P(1)$ must be false.

(b) TRUE. Since $P(29)$ is false, and there is no base case to start the induction. So $P(50)$ is not necessarily true. But there is nothing preventing $P(50)$ from being true, so it could either betrue or false.

(c) FALSE. Consider lists J_1: $C_1 > C_2$, J_2: $C_2 > C_1$, C_1: $J_1 > J_2$, C_2: $J_1 > J_2$

(d) TRUE. if J is the job at the top and its favorite candidate is C, then if (J,C') and (C,J') are in a pairing, (J,C) are a rogue couple because they mutually prefer each other.

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2. [Inequality.] Prove by induction on n that if n is a natural number and $x > 0$, then $(1+x)^n \geq 1+nx$.

Solution:

- Base case: When $n = 0$ the claim holds since $(1+x)^0 \geq 1+0\cdot x$.
- Inductive hypothesis: Now, assume as our inductive hypothesis that $(1+x)^k \geq 1+kx$ for some value of $n = k$ where $k > 0$.

- Inductive step: For $n = k+1$, we can show the following chain of inequalities:

$$
\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) && (1)\\
&\geq (1+kx)(1+x) \qquad \text{(by the inductive hypothesis)} && (2)\\
&\geq 1+kx+x+kx^2 && (3)\\
&\geq 1+(k+1)x+kx^2 && (4)\\
&\geq 1+(k+1)x, \qquad (kx^2 > 0 \text{ since } k > 0, x > 0) && (5)
\end{aligned}
$$

By induction, we have shown that $\forall n \in \mathbb{N}, (1+x)n \geq 1+nx$

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3. [Stable Matching.] Suppose that after running the Propose-and-Reject Algorithm with n jobs and n candidates, the pairing that results includes the pair $(1,A)$. Suppose that after a few days job 1 changes its mind, and decides that it does not like candidate A as much as it thought it did (i.e. it put her the last candidate on its preference list). What is the maximum number of rogue couples that result in the existing pairing from such a change to 1's preference list? Give a one or two sentence justification for why the number of rogue couples can be as large as you claim. Also give a one or two sentence justification for why the remaining couples cannot be rogue couples.

Solution: There can be $n-1$ rogue couples; this happens when all of the candidates prefer 1 to all the other jobs, and 1 at first preferred A to all other candidate but now puts her as its least preferred. This situation makes $(1,X)$ a rogue couple for all $X \neq A$. (i.e. one rogue couple for every candidate who is not A).

Moreover, there cannot be more than $n-1$ rogue couples. To see this, notice that whether or not (i,X) is a rogue couple can be determined just by looking at the preference lists of i and X (and the names of their partners). Since 1 is the only job whose preference list changes, it must be part of any rogue couple, and so there can be at most $n-1$ rogue couples (since $(1,A)$ clearly is not a rogue couple). So $n-1$ is the maximum number of rogue couples.

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1 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.

Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph G being a bipartite implies that G has no tours of odd length).

Solution:

Suppose there is a tour in the bipartite graph. Let us start traveling the tour from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the tour connects our start node n_0 to a node n_1 in R. The 2nd edge in the tour must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

2 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

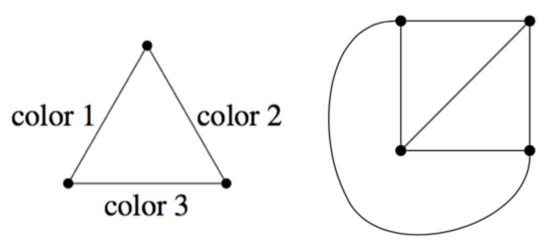
(a) Assume toward contradiction that $K_{3,3}$ were planar. In $K_{3,3}$, there are $v = 6$ vertices and $e = 9$ edges. If $K_{3,3}$ were planar, from Euler's formula we would have $v - e + f = 2 \Rightarrow f = 5$. On the other hand, each region is bounded by at least four edges, so $4f \leq 2e$, i.e., $20 \leq 18$, which is a contradiction. Thus, $K_{3,3}$ is not planar.

(b) In this problem, we use proof by contradiction. Assume G is planar. Select any five vertices out of the seven. Consider the subgraph formed by these five vertices. They cannot form K_5, since G is planar. So some pair of vertices amongst these five has no edge between them. Label these vertices v_1 and v_2. The remaining five vertices of G besides v_1 and v_2 cannot form K_5 either, so there is a second pair of vertices amongst these new five that has no edge between them. Label these v_3 and v_4. Label the remaining three vertices v_5, v_6 nd v_7. Since v_1v_2 is not an edge, by property T (which states any three vertices must have at least two edges between them)it must be that $\{v_1, v\}$ and $\{v_2, v\}$ are edges, where $v \in \{v_3, v_4, v_5, v_6, v_7\}$. Similarly for v_3, v_4 we have that $\{v_3, v\}$ and $\{v_4, v\}$ are edges, where $v \in \{v_1, v_2, v_5, v_6, v_7\}$. Now consider the subgraph induced by $\{v_1, v_2, v_3, v_5, v_6, v_7\}$. With the three vertices $\{v_1, v_2, v_3\}$ on one side and $\{v_5, v_6, v_7\}$ on the other, we observe that $K_{3,3}$ is a subgraph of this induced graph. This contradicts the fact that G is planar.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with greater than 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



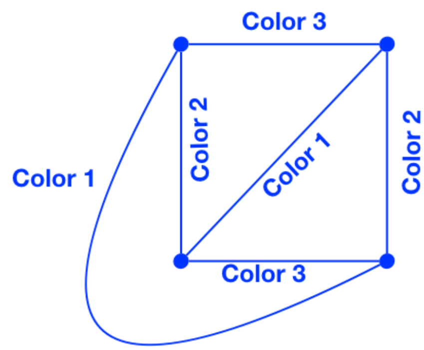
(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle. Now add the fourth vertex notice, call it vertex u. For any edge, say $\{u, v\}$ from this fourth vertex u, observe that the vertex v has two edges from before and hence there a third color available for the edge $\{u, v\}$.



(b) We will use induction on the number of edges n in the graph to prove the statement: If a graph G has $n \geq 0$ edges and the maximum degree of any vertex is d, then G can be colored with $2d-1$ colors.

Base case (n = 0). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.

Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.

Inductive step. Consider a graph G with $n = k+1$ edges. Remove an edge of your choice, say e from G. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d'-1$ colors and hence with $2d-1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts. This proves the statement for $n = k+1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices n in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is d, then the tree can be colored with d colors.

Base case (n=1). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.

Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.

Inductive Step. Remove any leaf v of your choice from the tree. We can then color the remaining tree with d colors by the inductive hypothesis. For any neighboring vertex u of vertex v, the degree of u is at most $d-1$ since we removed the edge $\{u,v\}$ along with the vertex v. Thus its incident edges use at most $d-1$ colors and there is a color available for coloring the edge {u,v}. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.