1 Short Answers - Graphs

(a) Bob removed a degree 3 node from an $n$-vertex tree. How many connected components are there in the resulting graph?

(b) Given an $n$-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph?

2 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property $T$: For every three distinct vertices $v_1, v_2, v_3$ of graph $G$, there are at least two edges among them. Use a proof by contradiction to show that if $G$ is a graph on $\geq 7$ vertices, and $G$ has property $T$, then $G$ is nonplanar.

3 Graph Coloring

Prove that a graph with maximum degree at most $k$ is $(k + 1)$-colorable.
4 Hypercubes

The vertex set of the $n$-dimensional hypercube $G = (V, E)$ is given by $V = \{0, 1\}^n$ (recall that $\{0, 1\}^n$ denotes the set of all $n$-bit strings). There is an edge between two vertices $x$ and $y$ if and only if $x$ and $y$ differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the $n$-dimensional hypercube is bipartite.