1 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.

(b) A simple graph obtained by adding an edge between two vertices of a tree creates a cycle.

(c) Adding an edge in a connected graph creates exactly one new cycle.
2 Coloring Trees

Prove that all trees with at least 2 vertices are bipartite: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[Hint: Use induction on the number of vertices.]
3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.

(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers 1, 2, 3 for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.
(c) Show that a tree can be edge colored with $d$ colors where $d$ is the maximum degree of any vertex.
4 Hypercubes

The vertex set of the $n$-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all $n$-bit strings). There is an edge between two vertices $x$ and $y$ if and only if $x$ and $y$ differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the $n$-dimensional hypercube is bipartite.