1 Banquet Arrangement

Suppose \( n \) people are attending a banquet, and each of them has at least \( m \) friends (\( 2 \leq m \leq n \)), where friendship is mutual. Prove that we can put at least \( m + 1 \) of the attendants on the same round table, so that each person sits next to his or her friends on both sides.

**Solution:** Let each person be a vertex and add an edge between two people if they are friends. Thus we have a graph with \( n \) vertices. Since each of them has at least \( m \) friends, we know that all the vertices in the graph have degree at least \( m \). Suppose we find a cycle of length at least \( m + 1 \) in this graph, say \( C = \{v_0, v_1, \ldots, v_k\} \), where \( k \geq m \). If we place these \( k + 1 \) people at the round table in the order given by the cycle \( C \), they observe that each person sits next to his or her friends since he/she has an edge with him/her in the corresponding graph. Thus we can rephrase the problem in graph theory terms as follows: given that all the vertices in an \( n \)-vertex graph have degree at least \( m \), show that there exists a cycle containing at least \( m + 1 \) vertices.

Let \( P = v_0v_1 \ldots v_l \) be a longest path in the graph. Such a path exists because the length of paths is bounded above by \( n \). All neighbors of \( v_0 \) must be in \( P \), since otherwise \( P \) can be extended to be even longer by appending this edge at the beginning of path \( P \). Let \( k \) be the maximum index of neighbors of \( v_0 \) along \( P \). Since \( v_0 \) has at least \( m \) neighbors, we must have \( k \geq m \). Then \( v_0v_1 \ldots v_kv_0 \) gives us the desired cycle.

2 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.

(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers 1, 2, 3 for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree \( d \geq 1 \) can be edge colored with \( 2d - 1 \) colors.
(c) Show that a tree can be edge colored with $d$ colors where $d$ is the maximum degree of any vertex.

**Solution:**

(a) Three color a triangle. Now add the fourth vertex notice, call it vertex $u$. For any edge, say $\{u, v\}$ from this fourth vertex $u$, observe that the vertex $v$ has two edges from before and hence there a third color available for the edge $\{u, v\}$.

(b) We will use induction on the number of edges $n$ in the graph to prove the statement: If a graph $G$ has $n \geq 0$ edges and the maximum degree of any vertex is $d$, then $G$ can be colored with $2d - 1$ colors.

Base case ($n = 0$). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.

Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.

Inductive step. Consider a graph $G$ with $n = k + 1$ edges. Remove an edge of your choice, say $e$ from $G$. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d' - 1$ colors and hence with $2d - 1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d - 1$ other edges, and thus at most $2(d - 1) = 2d - 2$ colors are unavailable for edge $e$. Thus, we can color edge $e$ without any conflicts. This proves the statement for $n = k + 1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices $n$ in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is $d$, then the tree can be colored with $d$ colors.

Base case ($n=1$). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.

Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.

Inductive Step. Remove any leaf $v$ of your choice from the tree. We can then color the remaining tree with $d$ colors by the inductive hypothesis. For any neighboring vertex $u$ of vertex $v$, the degree of $u$ is at most $d - 1$ since we removed the edge $\{u, v\}$ along with the vertex $v$. Thus its incident edges use at most $d - 1$ colors and there is a color available for coloring the edge.
{u,v}. This completes the inductive step and by induction we have that the statement holds for all \( n \geq 1 \).

3 Triangular Faces

Suppose we have a connected planar graph \( G \) with \( v \) vertices and \( e \) edges such that \( e = 3v - 6 \). Prove that in any planar drawing of \( G \), every face must be a triangle; that is, prove that every face must be incident to exactly three edges of \( G \).

Solution:
Suppose for the sake of contradiction that we have found a planar drawing of \( G \) such that one of the faces is incident on more than three edges. Choose an arbitrary vertex on that face to call \( v_0 \), and number the other vertices around the face \( v_1, v_2, ..., v_k \) proceeding clockwise from \( v_0 \). Since this face has at least 4 sides, we know that \( v_0 \) and \( v_2 \) do not have an edge between them. Furthermore, we know that we can add this edge to the planar drawing of \( G \) without having it cross any existing edges by just letting it cross the face. Thus, adding an edge between \( v_0 \) and \( v_2 \) results in a planar graph with \( v \) vertices and \( e + 1 = 3v - 5 \) edges. But we know that a planar graph can have at most \( 3v - 6 \) edges, so this is a contradiction. Thus, we must have that no such face exists; that is, we must have that every face in \( G \) is incident on exactly 3 edges.

4 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.
(b) Adding an edge between two vertices of a tree creates a new cycle.
(c) Adding an edge in a connected graph creates exactly one new cycle.
(d) We can create a soccer ball by stitching together 10 pentagons and 20 hexagonal pieces, with three pieces meeting at each vertex.

Solution:

(a) True.
Pick any pair of vertices \( x, y \). We know there is a path between them since the graph is
connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from \( x \) to \( y \). At some point (say at vertex \( a \)) the paths must diverge, and at some point (say at vertex \( b \)) they must reconnect. So by following the first path from \( a \) to \( b \) and the second path in reverse from \( b \) to \( a \) we get a cycle. This gives the necessary contradiction.

(b) **True.**

Pick any pair of vertices \( x \), \( y \) not connected by an edge. We prove that adding the edge \( \{x, y\} \) will create a cycle. From part (a), we know that there is a unique path between \( x \) and \( y \). Therefore, adding the edge \( \{x, y\} \) creates a cycle obtained by following the path from \( x \) to \( y \), then following the edge \( \{x, y\} \) from \( y \) back to \( x \).

(c) **False.**

In the following graph adding an edge creates two cycles.

(d) **False.**

If \( P \) pentagons and \( H \) hexagons are used, then there are \( f = P + H \) faces, \( v = (5P + 6H)/3 \) vertices, and \( e = (5P + 6H)/2 \) edges. Since a soccer ball is a polyhedron without holes, by Euler’s formula we have

\[
2 = v + f - e = \frac{5P + 6H}{3} + P + H - \frac{5P + 6H}{2} = \frac{P}{6}.
\]

Thus the number of pentagons must be 12 and not 10.