1. [True or False?] The following questions refer to stable matching instances with $n$ jobs and $n$ candidates.

(a) In any stable matching instance, in the pairing the Propose-and-Reject produces there is some job who gets their favorite candidate (the first candidate on their preference list).

(b) It is possible for a stable matching to have a job $A$ and a candidate $1$ be paired if $A$ is $1$’s least preferred choice and $1$ is $A$’s least preferred choice.

2. [True or False?]

(a) A graph with $k$ edges and $n$ vertices has a vertex of degree at least $2k/n$.

(b) If $e \leq 3v - 6$ holds for a graph $G$, then $G$ is planar.

(c) If all vertices of an undirected graph have degree 4, the graph must be the complete graph on five vertices, $K_5$.

3. [Coloring Trees]

Prove that all trees with at least 2 vertices are bipartite: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[Hint: Use induction on the number of vertices.]
1 Baby Fermat

Assume that $a$ does have a multiplicative inverse mod $m$. Let us prove that its multiplicative inverse can be written as $a^k \pmod{m}$ for some $k \geq 0$.

(a) Consider the sequence $a, a^2, a^3, \ldots \pmod{m}$. Prove that this sequence has repetitions. 
   (Hint: Consider the Pigeonhole Principle.)

(b) Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what can you say about $a^{i-j} \pmod{m}$?

(c) Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is $k$ in terms of $i$ and $j$?

2 Bijections

Let $n$ be an odd number. Let $f(x)$ be a function from $\{0, 1, \ldots, n-1\}$ to $\{0, 1, \ldots, n-1\}$. In each of these cases say whether or not $f(x)$ is necessarily a bijection. Justify your answer (either prove $f(x)$ is a bijection or give a counterexample).

(a) $f(x) = 2x \pmod{n}$.

(b) $f(x) = 5x \pmod{n}$.

(c) $n$ is prime and

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^{-1} \pmod{n} & \text{if } x \neq 0. \end{cases}$$

(d) $n$ is prime and $f(x) = x^2 \pmod{n}$. 

3 Introduction to Chinese Remainder Theorem

Solve for $x \in \mathbb{Z}$ where

$$
\begin{align*}
x &\equiv 3 \pmod{11}, \\
x &\equiv 7 \pmod{13}.
\end{align*}
$$

(a) Find the multiplicative inverse of 13 modulo 11.

(b) What is the smallest $b \in \mathbb{Z}^+$ such that $13 \mid b$ and $b \equiv 3 \pmod{11}$?

(c) Find the multiplicative inverse of 11 modulo 13.

(d) What is the smallest $a \in \mathbb{Z}^+$ such that $11 \mid a$ and $a \equiv 7 \pmod{13}$?

(e) Now, write down the set of possible solutions for $x$. 

CS 70, Spring 2020, DIS 3B