

1. [True or False?] The following questions refer to stable matching instances with n jobs and n candidates.

- (a) In any stable matching instance, in the pairing the Propose-and-Reject produces there is some job who gets their favorite candidate (the first candidate on their preference list).
- (b) It is possible for a stable matching to have a job A and a candidate 1 be paired if A is 1 's least preferred choice and 1 is A 's least preferred choice.

2. [True or False?]

- (a) A graph with k edges and n vertices has a vertex of degree at least $2k/n$.
- (b) If $e \leq 3v - 6$ holds for a graph G , then G is planar.
- (c) If all vertices of an undirected graph have degree 4, the graph must be the complete graph on five vertices, K_5 .

3. [Coloring Trees]

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint*: Use induction on the number of vertices.]

1 Baby Fermat

Assume that a does have a multiplicative inverse mod m . Let us prove that its multiplicative inverse can be written as $a^k \pmod{m}$ for some $k \geq 0$.

(a) Consider the sequence $a, a^2, a^3, \dots \pmod{m}$. Prove that this sequence has repetitions. (**Hint:** Consider the Pigeonhole Principle.)

(b) Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what can you say about $a^{i-j} \pmod{m}$?

(c) Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is k in terms of i and j ?

2 Bijections

Let n be an odd number. Let $f(x)$ be a function from $\{0, 1, \dots, n-1\}$ to $\{0, 1, \dots, n-1\}$. In each of these cases say whether or not $f(x)$ is necessarily a bijection. Justify your answer (either prove $f(x)$ is a bijection or give a counterexample).

(a) $f(x) = 2x \pmod{n}$.

(b) $f(x) = 5x \pmod{n}$.

(c) n is prime and

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^{-1} \pmod{n} & \text{if } x \neq 0. \end{cases}$$

(d) n is prime and $f(x) = x^2 \pmod{n}$.

3 Introduction to Chinese Remainder Theorem

Solve for $x \in \mathbb{Z}$ where

$$\begin{aligned}x &\equiv 3 \pmod{11}, \\x &\equiv 7 \pmod{13}.\end{aligned}$$

- (a) Find the multiplicative inverse of 13 modulo 11.

- (b) What is the smallest $b \in \mathbb{Z}^+$ such that $13 \mid b$ and $b \equiv 3 \pmod{11}$?

- (c) Find the multiplicative inverse of 11 modulo 13.

- (d) What is the smallest $a \in \mathbb{Z}^+$ such that $11 \mid a$ and $a \equiv 7 \pmod{13}$?

- (e) Now, write down the set of possible solutions for x .