CS 70 Spring 2021 — Discrete Mathematics and Probability Theory — DIS 5B

1 Clothing Argument

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(b) How many outfits are there if we wanted to wear exactly two categories?

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

Solution:

(a) 10^4 by the first rule of counting.

(b) $\binom{4}{2} \cdot 10^2$ First we choose the two categories, then we apply first rule of counting.

(c) $\binom{10}{4} \cdot 4! = \frac{10!}{6!}$ We choose the 4 hats and then multiply by the number of ways to rearrange them. This is 10 permutation 4 (although that notation was not covered in this class I don't think, so don't worry about that)

(d) $\binom{10}{4}$ or written as a function of the previous part, $c/4!$.

2 Strings

What is the number of strings you can construct given:

(a) n ones, and m zeroes?

(b) n_1 A's, n_2 B's and n_3 C's?

(c) $n_1, n_2, \ldots, n_k$ respectively of k different letters?

Solution:

(a) $\binom{n+m}{n}$

(b) $(n_1+n_2+n_3)!/(n_1!\cdot n_2!\cdot n_3!)$

(c) $(n_1+n_2+\cdots+n_k)!/(n_1!\cdot n_2!\cdots n_k!)$.

3 Counting Game

RPG games are all about explore different mazes. Here is a weird maze: there are 2^n rooms, where each room is the vertex on a the n-dimensional hypercube, labeled by a n bit binary string.

For each room, there are n different doors, each door corresponding to an edge on the hypercube. If you are at room i, and choose door j, then you will go to room $i \oplus 2^j$ (flips the $j+1$-th bit in number i).

(a) How many different shortest path are there from room 0 to room 2^n-1?

(b) How many different paths of $n+2$ steps are there to go from room 0 to room 2^n-1?

(c) If $n=8$, how many different shortest pathes are there from room 0 to room 63 that pass through 3 and 19?

Solution:

(a) $n!$, the shortest path is n, and for the i-th step, there are only $n-i$ doors flips a zero to one.

(b) **Solution I:** The player made one mistake during his trip, so suppose he made the mistake at step i, $i>0$, so there are i different ways to make the mistake. Then he will start from a room with $n-i+1$ zeros. So the total number is $\sum_{i=1}^{n}(\binom{n}{i}*i!*i*(n-i+1)!)$.

Optional for further steps:

$\sum_{i=1}^{n}(\binom{n}{i}*i!*i*(n-i+1)!) = \sum_{i=1}^{n}\frac{n!*i!*i*(n-i+1)!}{(n-i)!i!)} = \sum_{i=1}^{n} n!*i*(n-i+1) = n!\sum_{i=1}^{n} i(n-i+1) = n!*(\sum_{i=1}^{n}(in-i^2+i))$

To calculate the second term: $\sum_{i=1}^{n}(in-i^2+i) = n*\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(n+2)}{6}$

Putting it together, the simplified answer is $n!\cdot\frac{n(n+1)(n+2)}{6} = \frac{(n+2)!n}{6}$.

Solution II: The player made one mistake during his trip, so exactly one bit is flipped three times (the bit corresponding to the mistake), with the rest of them being flipped once. Then, the problem becomes counting the number of ways to order $n+2$ bit flips, where 3 of the flips are identical. Therefore, the number of such paths is $\frac{(n+2)!n}{6}$, since there are n ways to choose the bit which is flipped three times and $\frac{(n+2)!}{6}$ ways to arrange the flips.

(c) From 0 to 3, 2 different paths. From 3 to 19: notice $3\oplus 19=16$ so there is only one way. From 19 to 63, there are 3 zeros in $63\oplus 19$ so total 3! different paths. In total $2*3!$ different paths.