CS 70 Discrete Mathematics and Probability Theory
Spring 2020

DIS 6A

1 Clothing Argument

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(b) How many outfits are there if we wanted to wear exactly two categories?

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

(e) Now we've packed four distinct hats. Suppose we are on a 3-day trip and need to pick a hat to wear every day (we can wear the same hat for multiple days). How many different hat-wearing schedules are there?

Solution:

(a) 10^4

(b) $\binom{4}{2} \cdot 10^2$

(c) $\binom{10}{4} \cdot 4! = \dfrac{10!}{6!}$

(d) $\binom{10}{4}$ or written as a function of the previous part, $c/4!$.

(e) 4^3

2 Counting Practice

(a) If you shuffle two (identical) decks of cards together, you get a stack of 104 cards, where each different card type is included twice. How many different ways are there to order this stack of cards?

(b) How many different anagrams of GHOST are there if: (1) H is the right neighbor of G; (2) G is on the left of H (and not necessarily H's neighbor)?

(c) There are 20 socks in a drawer, none of which match. How many different ways are there to pair up these socks? (Assume that any sock can be paired with any other sock.)

Solution:

(a) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $104!/2^{52}$.

(b) (1) We consider GH is a new letter $\mathscr{G}$, then the question becomes counting the rearranging of 4 distinct letters, and is 4!. (2) Let A be the set of all the rearranging of GHOST with G on the left side of H, and B be the set of all the rearranging of GHOST with G on the right side of H. $|A \bigcup B| = 5!, |A \cap B| = 0$. There is a bijection between A and B by constructing an operation that exchanges the positions of G and H. Thus $|A| = |B| = 5!/2$.

(c) **Answer 1:** Let's number the socks from 1 to 20. Sock 1 has 19 choices for its mate. Let i be the smallest index among socks that have not yet been assigned a mate. Then no matter what the value of i is (in particular, i could be 2 or 3), sock i has 17 choices for its mate. The next smallest indexed sock that doesn't have a mate now has 15 choices for its mate. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the socks numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the socks at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the socks doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of socks. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any sock pair (x, y), sock x could have appeared in position $2i-1$ and sock y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $20!/(10! \cdot 2^{10})$ pairings.

Answer 3: In the first step, pick a pair of socks from the 20 socks. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of socks from the remaining 18 socks. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of socks from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 socks, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, $\dots$, tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\dots\binom{2}{2}/10!$ different pairings of 20 socks.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

3 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

Solution:

One counting strategy is based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10?5?1 = 4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$. possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.

4 Counting on Graphs

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

Solution:

(a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^{m}\binom{m}{k}$, where $m = n(n-1)/2$ in this case.)

(b) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

(c) Without considering symmetries there are 6! ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.