1 Warm-up

For each of the following parts, you may leave your answer as an expression.

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d. and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of $X$.

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3 cars per week. Calculate the probability that there is at least one accident this week.

(d) Consider an experiment that consists of counting the number of $\alpha$ particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such $\alpha$-particles are given off per second, what is a good approximation to the probability that no more that 2 $\alpha$-particles will appear in a second?
2 Coupon Collector Variance

It’s that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of \( n \) different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let \( X \) be the number of visits you have to make before you can redeem the grand prize. Show that
\[
\text{Var}(X) = n^2 \left( \sum_{i=1}^{n} i^{-2} \right) - \mathbb{E}(X).
\]

[Hint: Try to express the number of visits as a sum of geometric random variables as with the coupon collector’s problem. Are the variables independent?]
3 Boutique Store

Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model $X$, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean $\lambda$.

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability $p$. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as $Y$ and the number of them that do not buy anything as $Z$ (so $X = Y + Z$).

(a) What is the probability that $Y = k$ for a given $k$? How about $P[Z = k]$? Hint: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$ 

(b) State the name and parameters of the distribution of $Y$ and $Z$.

(c) Prove that $Y$ and $Z$ are independent. In particular, prove that for every pair of values $y, z$, we have $P[Y = y, Z = z] = P[Y = y]P[Z = z]$. 