CS 70 Discrete Mathematics and Probability Theory
Spring 2020 DIS 6B

1 Farmer's Market

Suppose you want k items from the farmer's market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

(b) There are pumpkins, apples, oranges, and pears at the market

(c) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

Solution:

This is a classic "balls and bins" (also known as "stars and bars") problem.

(a) $k+1$.

(b) $\binom{k+3}{3}$.

(c) There are $\binom{n+k-1}{n-1}$ ways to choose k fruits of n types with no additional restrictions. n of these combinations, however, contain only one variety of fruit, so we subtract them for a total of $\binom{n+k-1}{n-1} - n$.

2 Charming Star

At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same. A "voting combination" is defined by how many votes each candidate receives. In this question, only the number of votes for each candidate matters; it does not matter which specific people voted for each candidate.

(a) How many possible voting combinations are there for the 5 candidates?

(b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?

Solution:

(a) Let x_i be the number of votes of the i-th candidates. We would like to find all possible combinations of $(x_1, x_2, x_3, x_4, x_5)$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100.$$

It is equivalent to selecting $k = 100$ objects from $n = 5$ categories. The number of possible combinations is:

$$\binom{100+5-1}{100} = \binom{104}{100} = 4598126.$$

(b) Now we have a constraint that one of the x_i should be at least 51. Say, let x_1 be at least 51. It is equivalent to giving the first candidate 51 votes at the beginning and then distributing the remaining 49 votes to them again. The number of possible combinations is $\binom{49+5-1}{49}$. Since one of the 5 candidates could have at least 51 votes, the total number of possible voting combinations such that exactly one candidate gets more than 50 votes is:

$$\binom{5}{1}\binom{49+5-1}{49} = 1464125.$$

3 Story Problems

Prove the following identities by combinatorial argument:

(a) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

(b) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(c) $n^2 = 2\binom{n}{2} + n$

(d) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$
Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").

(e) $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$
Hint: Consider a generalization of the previous part.

Solution:

(a) The left hand side is the number of ways to choose k elements out of n. Looking at this another way, we look at the first element and decide whether we are going to choose it or not. If we choose it, then we need to choose $k-1$ more elements from the remaining $n-1$. If we don't choose it, then we need to choose all our k elements from the remaining $n-1$. We are not double counting, since in one of our cases we chose the first element and in the other, we did not.

(b) The left hand side is the number of ways to choose two elements out of $2n$. Counting in another way, we first divide the $2n$ elements (arbitrarily) into two sets of n elements. Then we consider three cases: either we choose both elements out of the first n-element set, both out of the second n-element set, or one element out of each set. The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these three cases are mutually exclusive and cover all the possibilities, summing them must give the same number as the left hand side. This completes the proof.

(c) LHS: There are n movies. Choose the best-rated movie and the best-selling movie. There are n choices for each title.

RHS: Choose 2 distinct movies, permute them in 2 ways. But what if the best-rated movie is the best selling movie? Then we're just choosing one movie, and there are n ways of doing that.

(d) RHS: From n people, pick one team-leader and some (possibly empty) subset of other people on his team.

LHS: First pick k people on the team, then pick the leader among them.

(e) RHS: Form a team as follows: Pick j leaders from n people. Then pick some (possibly empty) subset of the remaining people.

LHS: First pick $k \geq j$ people on the team, then pick the j leaders among them.