1. [True or False]

(a) The set of all irrational numbers \( \mathbb{R} \setminus \mathbb{Q} \) (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers \( x \) that solve the equation \( 3x \equiv 2 \pmod{10} \) is countably infinite.

(c) The set of real solutions for the equation \( x + y = 1 \) is countable.

For any two functions \( f : Y \to Z \) and \( g : X \to Y \), let their composition \( f \circ g : X \to Z \) be given by \( f \circ g = f(g(x)) \) for all \( x \in X \). Determine if the following statements are true or false.

(d) \( f \) and \( g \) are injective (one-to-one) \( \implies \) \( f \circ g \) is injective (one-to-one).

(e) \( f \) is surjective (onto) \( \implies \) \( f \circ g \) is surjective (onto).

Solution:

(a) True. Proof by contradiction. Suppose the set of irrationals is countable. From Lecture note 10 we know that the set \( \mathbb{Q} \) is countable. Since union of two countable sets is countable, this would imply that the set \( \mathbb{R} \) is countable. But again from Lecture note 10 we know that this is not true. Contradiction!

(b) True. Multiplying both sides of the modular equation by 7 (the multiplicative inverse of 3 with respect to 10) we get \( x \equiv 4 \pmod{10} \). The set of all integers that solve this is \( S = \{10k + 4 : k \in \mathbb{Z}\} \) and it is clear that the mapping \( k \in \mathbb{Z} \) to \( 10k + 4 \in S \) is a bijection. Since the set \( \mathbb{Z} \) is countably infinite, the set \( S \) is also countably infinite.

(c) False. Let \( S \in \mathbb{R} \times \mathbb{R} \) denote the set of all real solutions for the given equation. For any \( x' \in \mathbb{R} \), the pair \( (x', y') \in S \) if and only if \( y' = 1 - x' \). Thus \( S = \{(x, 1-x) : x \in \mathbb{R}\} \). Besides, the mapping \( x \) to \( (x, 1-x) \) is a bijection from \( \mathbb{R} \) to \( S \). Since \( \mathbb{R} \) is uncountable, we have that \( S \) is uncountable too.

(d) True. Recall that a function \( h : A \to B \) is injective iff \( a_1 \neq a_2 \implies h(a_1) \neq h(a_2) \) for all \( a_1, a_2 \in A \). Let \( x_1, x_2 \in X \) be arbitrary such that \( x_1 \neq x_2 \). Since \( g \) is injective, we have \( g(x_1) \neq g(x_2) \). Now, since \( f \) is injective, we have \( f(g(x_1)) \neq f(g(x_2)) \). Hence \( f \circ g \) is injective.

(e) False. Recall that a function \( h : A \to B \) is surjective iff \( \forall b \in B, \exists a \in A \) such that \( h(a) = b \). Let \( g : \{0, 1\} \to \{0, 1\} \) be given by \( g(0) = g(1) = 0 \). Let \( f : \{0, 1\} \to \{0, 1\} \) be given by \( f(0) = 0 \) and \( f(1) = 1 \). Then \( f \circ g : \{0, 1\} \to \{0, 1\} \) is given by \( (f \circ g)(0) = (f \circ g)(1) = 0 \). Here \( f \) is surjective but \( f \circ g \) is not surjective.
2. Consider an \( n \times n \) matrix \( A \) where the diagonal consists of alternating 1’s and 0’s starting from 1, i.e. \( A[0,0] = 1, A[1,1] = 0, A[2,2] = 1, \) etc. Describe an \( n \) length vector from \( \{0,1\}^n \) that is not equal to any row in the matrix \( A \). (Note that the all ones vector or the all zeros vector of length \( n \) could each be rows in the matrix.)

**Solution:**

The row consisting of alternating 1’s and 0’s starting with 0. That is, 010101… Recall the diagonalization idea of constructing something not in a list by making it different along the diagonal.

3. Find the precise error in the following proof:

**False Claim:** The set of rationals \( r \) such that \( 0 \leq r \leq 1 \) is uncountable.

**Proof:** Suppose towards a contradiction that there is a bijection \( f : \mathbb{N} \rightarrow \mathbb{Q}[0,1] \), where \( \mathbb{Q}[0,1] \) denotes the rationals in \( [0,1] \). This allows us to list all the rationals between 0 and 1, with the \( j \)-th element of the list being \( f(j) \). Suppose we represent each of these rationals by their non-terminating expansion (for example, 0.4999… rather than 0.5). Let \( d_j \) denote the \( j \)-th digit of \( f(j) \). We define a new number \( e \), whose \( j \)-th digit \( e_j \) is equal to \( (d_j + 2) \pmod{10} \). We claim that \( e \) does not occur in our list of rationals between 0 and 1. This is because \( e \) cannot be equal to \( f(j) \) for any \( j \), since it differs from \( f(j) \) in the \( j \)-th digit by more than 1. Contradiction. Therefore the set of rationals between 0 and 1 is uncountable.

**Solution:** The number \( e \) constructed in the proof can be irrational (in fact, it has to be irrational). And hence \( e \) being different from all the numbers \( f(j) \) does not lead to a contradiction.