CS 70 Spring 2020 | Discrete Mathematics and Probability Theory Course Notes | DIS 07B

1 Monty Hall Challenge

Let us take on the challenge posed in lecture, and formally analyze the Monty Hall Problem.

(a) Assume that the corgi (the prize) and two goats were placed uniformly at random behind the three doors. What is the probability space $(\Omega, \mathbb{P})$?

(b) If our contestant chose door 1 in the first round, and decides to switch to another door after being shown a goat behind door 2 or 3, what are the events C_1 ="They win the corgi" and $\overline{C_1}$ ="They win a goat"? What are their probabilities $\mathbb{P}(C_1)$ and $\mathbb{P}(\overline{C_1})$?

(c) If the contestant does not switch doors, what are the events $C_2, \overline{C_2}$ of winning the corgi and goats, and their respective probabilities now?

(d) If instead of choosing door 1 in the beginning, they chose a door uniformly at random, how do your $\Omega, \mathbb{P}, C_i, \overline{C_i}$ from above change?

Solution:

(a) The randomness here lies in how the animals were distributed behind the doors. The possible outcomes are collected in the sample space $\Omega = \{CGG, GCG, GGC\}$, where each sequence encodes what animal hides behind which door, e.g. CGG means the corgi is behind door 1, and the goats behind doors 2 and 3. Since we are placing animals uniformly, the probability $\mathbb{P}(\omega)$ of each outcome ω is $1/|\Omega| = 1/3$.

(b) If the corgi sleeps behind door 1, then the contestant can only win a goat after switching. If, however, a goat is behind door 1, then the contestant will always win the corgi after switching, since Carol shows him the other goat! So $C_1 = \{GCG, GGC\}$, while $\overline{C_1} = \Omega \setminus C_1 = \{CGG\}$. As a result, the associated probabilites are $\mathbb{P}(C_1) = 2/3, \mathbb{P}(\overline{C_1}) = 1/3$.

(c) Now the roles of C_1 and $\overline{C_1}$ invert: If the contestant does not switch doors, they can only win if the corgi is behind door 1, i.e. $C_2 = \{CGG\}$ and $\overline{C_2} = \{GCG, GGC\}$. So $\mathbb{P}(C_2) = 1/3, \mathbb{P}(\overline{C_2}) = 2/3$.

(d) Our new sample space Ω' now becomes bigger since the outcomes include the choice of our contestant: $\Omega' = \{1,2,3\} \times \Omega$, where for any element $(i,s) \in \Omega'$, i indicates the choice of door, and s is a sequence of animals as before. Since everything is equally likely, individual probabilites are now $\mathbb{P}(\omega) = 1/|\Omega'| = 1/9$. Regardless of the choice i however, there are still two outcomes in which the contestant wins if he switches, and only one if he doesn't switch.

So $|C_1| = 2\cdot 3 = 6$ and $|C_2| = 1\cdot 3 = 3$, yielding overall probabilities $\mathbb{P}(C_1) = 2/3, \mathbb{P}(C_2) = 1/3$. We could have phrased this in terms of the law of total probability to obtain the same result: $\mathbb{P}(C_1) = \sum_{i=1}^{3} \mathbb{P}(E_i \cap C_1) = \sum_{i=1}^{3} \mathbb{P}(E_i)\mathbb{P}(C_1 \mid E_i) = \sum_{i=1}^{3} \frac{1}{3}\cdot\frac{2}{3}$, where E_i is the event of our contestant picking door i as their first choice.

2 Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. [*Hint*: The answer is NOT 1/2.]

Solution:

(a) Each flip results in either heads (H) or tails (T). So in total the total number of outcomes is 8, which we represent by length 3 strings of H's and T's. We have

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(b) These are the strings that start with H. We have $A = \{HHH, HHT, HTH, HTT\}$. There are 4 such outcomes.

(c) These are the strings that end with an H. We have $B = \{HHH, HTH, THH, TTH\}$. There are 4 such outcomes.

(d) These are the strings that start and end with an H. We have $C = \{HHH, HTH\}$. There are 2 such outcomes.

(e) We have $D = \{HHH, HHT, HTH, HTT, THH, TTH\}$. There are 6 such outcomes.

(f) No, A and B are not disjoint. For example HHH belongs to both of them.

The event C is the intersection of A and B, because in C we require exactly both A (the first coin being heads) and B (the third coin being heads) to happen. So $C = A \cap B$.

The event D is the union of A and B, because in D we require at least one of A (the first coin being heads) or B (the second coin being heads) to happen. So $D = A \cup B$.

(g) First, obviously $|\Omega| = 2^n$.

Note that for each outcome in the three-coin case, there are now 2^{n-3} outcomes, each corresponding to a possible configuration of the 4th flip and beyond. Since A, B, C, and D do not care about the outcomes of the 4th flip and beyond, this means that the size of each set is simply multiplied by 2^{n-3}. Therefore we have $|A| = 4 \times 2^{n-3} = 2^{n-1}$, $|B| = 4 \times 2^{n-3} = 2^{n-1}$, $|C| = 2 \times 2^{n-3} = 2^{n-2}$, and $|D| = 6 \times 2^{n-3} = 3 \times 2^{n-2}$.

(h) There are 6 possible outcomes which are all equally likely. We have 3 choices for the coin that we draw (which we represent by HH, HT and TT). Then for each coin we have two choices, we either see the first side or the second side (which we represent by 1 and 2). So the outcomes are $\Omega = \{(HH,1),(HH,2),(HT,1),(HT,2),(TT,1),(TT,2)\}$. Now given that we saw a heads, we can get rid of 3 of the outcomes, and the possible remaining outcomes are $\{(HH,1),(HH,2),(HT,1)\}$ which are all equally likely. In this space, the event that the coin has two heads is $\{(HH,1),(HH,2)\}$ which consists of two equally likely outcomes. So the probability is $2/3$.

3 Sampling

Suppose you have balls numbered $1,\ldots,n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

Solution:

(a) Out of n^2 pairs of balls that you could have chosen, only one pair $(1,2)$ corresponds to the event we are interested in, so the probability is $1/n^2$.

(b) Again, there are n^2 total outcomes. Now, we want to count the number of outcomes where the second ball's number is strictly less than the first ball's number. Similarly to the last part, we can view any outcome as an ordered pair (n_1,n_2), where n_1 is the number on the first ball, and n_2 is the number on the second ball. There are $\binom{n}{2}$ outcomes where $n_1 > n_2$; select two distinct numbers from $[1,n]$, and assign the higher number to n_1. Thus, the probability is $\frac{\binom{n}{2}}{n^2} = \frac{n-1}{n^2}$.

Alternate Solution: The probability that the two balls have the same number is $n/n^2 = 1/n$, so the probability that the balls have different numbers is $1 - 1/n = (n-1)/n$. By symmetry, it is equally likely for the first ball to have a greater number and for the second ball to have a greater number, so we take the probability $(n-1)/n$ and divide it by two to obtain $(n-1)/(2n)$.

(c) Again, there are n^2 pairs of balls that we could have drawn, but there are $n-1$ pairs of balls which correspond to the event we are interested in: $\{(1,2),(2,3),\dots,(n-1,n)\}$. So, the probability is $(n-1)/n^2$.

(d) There are a total of $n(n-1)$ pairs of balls that we could have drawn, and only the pair $(1,2)$ corresponds to the event that we are interested in, so the probability is $1/(n(n-1))$.

The probability that the two balls are the same is now 0, but the symmetry described earlier still applies, so the probability that the second ball has a smaller number is $1/2$.

There are a total of $n(n-1)$ pairs of balls that we could have drawn, and we are interested in the $n-1$ pairs $(1,2),(2,3),\dots,(n-1,n)$ as before. Thus, the probability that the second ball is one greater than the first ball is $1/n$.