1. Diagonalization

How many different ways are there to rearrange the letters of DIAGONALIZATION (15 letters with 3 A’s, 3 I’s, 2 N’s, and 2 O’s) without the two N’s being adjacent?

Solution:
\[
\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!} = 13 \cdot 14!
\]

The word DIAGONALIZATION has 15 letters with 3 A’s, 3 I’s, 2 N’s, and 2 O’s, so there are \(\frac{15!}{3!3!2!2!}\) ways to rearrange the letters in total.

The number of rearrangements where the two N’s are adjacent is \(\frac{14!}{3!3!2!}\), where we have considered "NN" as a single character. The difference \(\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!}\) is then equal to the number of rearrangements without the two N’s being adjacent.

2. Jelly Bean Factory

A candy factory has an endless supply of red, orange and yellow jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each, with each possible combination of colors in the jar being equally likely. (One possible color combination, for example, is a jar of 56 red, 22 orange, and 22 yellow jelly beans.)

Find \(N\), the number of different possible color combinations of jelly beans in a single jar (note that color combinations are unordered).

Solution: The correct way to think of this problem is in terms of 100 unlabeled balls (the jelly beans) in 3 labeled bins (the colors red, orange, yellow). Recall from class that we have a formula for this: \(\binom{n+k-1}{k}\), where \(k\) is the number of balls and \(n\) is the number of bins. Therefore, the correct answer is

\[
N = \binom{102}{100} = 5151
\]

3. Combinatorial Proof I

Provide a combinatorial proof that \(\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}\)

Solution:
LHS: first choose a group of \(r\) people from \(n\) people, then choose \(k\) leaders among those \(r\).
RHS: first choose $k$ leaders, then among the rest $(n - k$ people), choose $r - k$ followers to form a group of size $r$ ($r - k + k = r$).

4 Permutations

(a) How many permutations of the numbers 1 through $n$ are there?

(b) How many permutations of the numbers 1 through $n$ are there such that 1 comes before 2 and after 3? (Assume $n > 3$)

(c) For each permutation $\sigma$ of 1 through $n$, let $\sigma(i)$ denote the value at position $i$. For example, if the permutation is 2, 4, 1, 3, we have $\sigma(1) = 2$ and $\sigma(2) = 4$.

For a fixed $1 \leq k \leq n$, how many permutations $\sigma$ of 1 through $n$ are there where for all $i < k$, $\sigma(i) < \sigma(k)$? (Express your answer in terms of $n$ and $k$.)

Solution:

(a) $n!$

(b) $n!/6$. $\binom{n}{3}$ ways to pick positions for 1, 2, and 3, and $(n - 3)!$ ways to permute the remaining objects.

(c) $n!/k$. $1/k$ of the permutations have $\sigma(k)$ being the largest of the first $k$ elements in $\sigma$. 

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