1. The Count

How many of the first 100 positive integers are divisible by 2, 3, or 5?

**Solution:**

We use inclusion-exclusion to calculate the number of numbers that satisfy this property. Let $A$ be the set of numbers divisible by 2, $B$ be the set of numbers divisible by 3, and $C$ be the set of numbers divisible by 5. Then, we calculate

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{6} \right\rfloor - \left\lfloor \frac{100}{10} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor + \left\lfloor \frac{100}{30} \right\rfloor$$

$$= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

numbers.

2. Two Proof Methods

Consider the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

1. Prove the identity by algebraic manipulation (using the formula for the binomial coefficients).

2. Prove the identity using a combinatorial argument.

**Solution:**

1. 

$$\binom{2n}{2} = \frac{(2n)!}{2!(2n-2)!}$$

$$= \frac{2n(2n-1)}{2}$$

$$= n(2n-1)$$

$$= n(n-1) + n^2$$

$$= \frac{2n(n-1)}{2} + n^2$$

$$= 2\binom{n}{2} + n^2.$$
2. The left hand side is the number of ways to choose two elements out of $2n$ (say, choosing two representatives of a group of $2n$ people). Counting in another way, we first divide the $2n$ elements (arbitrarily) into two sets of $n$ elements (say, divide the group into two teams of $n$ people, team $A$ and team $B$). Then we consider three cases: either we choose both elements out of the first $n$-element set (both representatives are from team $A$), both out of the second $n$-element set (both representatives are from team $B$), or one element out of each set (one representative is from team $A$, and one is from team $B$). The number of ways we can do each of these things is \( \binom{n}{2} \), \( \binom{n}{2} \), and $n^2$, respectively. Since these three cases are mutually exclusive and cover all the possibilities, summing them must give the same number as the left hand side. This completes the proof.

Comment: To see why picking one element from each set is $n^2$, we see that for each choice (each team $A$ member) from the first set, there are $n$ choices (team $B$ members) from the second set. And since there are $n$ elements in the first set, by the Product Rule, the total number of ways to do this is \( n \cdot n = n^2 \).

3 Monty Hall Challenge

Let us take on the challenge posed in lecture, and formally analyze the Monty Hall Problem.

(a) Assume that the corgi (the prize) and two goats were placed uniformly at random behind the three doors. What is the probability space \( (\Omega, \mathbb{P}) \)?

(b) If our contestant chose door 1 in the first round, and decides to switch to another door after being shown a goat behind door 2 or 3, what are the events \( C_1 = \text{"They win the corgi"} \) and \( \overline{C_1} = \text{"They win a goat"} \)? What are their probabilities \( \mathbb{P}(C_1) \) and \( \mathbb{P}(\overline{C_1}) \)?

(c) If the contestant does not switch doors, what are the events \( C_2, \overline{C_2} \) of winning the corgi and goats, and their respective probabilities now?

(d) If instead of choosing door 1 in the beginning, they chose a door uniformly at random, how do your \( \Omega, \mathbb{P}, C_i, \overline{C_i} \) from above change?

Solution:

(a) The randomness here lies in how the animals were distributed behind the doors. The possible outcomes are collected in the sample space \( \Omega = \{CGG, GCG, GGC\} \), where each sequence encodes what animal hides behind which door, e.g. \( CGG \) means the corgi is behind door 1, and the goats behind doors 2 and 3. Since we are placing animals uniformly, the probability \( \mathbb{P}(\omega) \) of each outcome \( \omega \) is \( 1/|\Omega| = 1/3 \).

(b) If the corgi sleeps behind door 1, then the contestant can only win a goat after switching. If, however, a goat is behind door 1, then the contestant will always win the corgi after switching, since Carol shows him the other goat! So \( C_1 = \{GCG, GGC\} \), while \( \overline{C_1} = \Omega \setminus C_1 = \{CGG\} \). As a result, the associated probabilities are \( \mathbb{P}(C_1) = 2/3, \mathbb{P}(\overline{C_1}) = 1/3 \).
(c) Now the roles of $C_1$ and $\bar{C}_1$ invert: If the contestant does not switch doors, they can only win if the corgi is behind door 1, i.e. $C_2 = \{CGG\}$ and $\bar{C}_2 = \{GCG, GGC\}$. So $\mathbb{P}(C_2) = 1/3, \mathbb{P}(\bar{C}_2) = 2/3$.

(d) Our new sample space $\Omega'$ now becomes bigger since the outcomes include the choice of our contestant: $\Omega' = \{1, 2, 3\} \times \Omega$, where for any element $(i, s) \in \Omega'$, $i$ indicates the choice of door, and $s$ is a sequence of animals as before. Since everything is equally likely, individual probabilities are now $\mathbb{P}(\omega) = 1/|\Omega'| = 1/9$. Regardless of the choice $i$ however, there are still two outcomes in which the contestant wins if he switches, and only one if he doesn’t switch. So $|C_1| = 2 \cdot 3 = 6$ and $|C_2| = 1 \cdot 3 = 3$, yielding overall probabilities $\mathbb{P}(C_1) = 2/3, \mathbb{P}(C_2) = 1/3$. We could have phrased this in terms of the law of total probability to obtain the same result: $\mathbb{P}(C_1) = \sum_{i=1}^{3} \mathbb{P}(E_i \cap C_1) = \sum_{i=1}^{3} \mathbb{P}(E_i) \mathbb{P}(C_1 | E_i) = \sum_{i=1}^{3} \frac{1}{3} \cdot \frac{2}{3}$, where $E_i$ is the event of our contestant picking door $i$ as their first choice.

4 Sampling

Suppose you have balls numbered 1, . . . , $n$, where $n$ is a positive integer $\geq 2$, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball’s number is strictly less than the first ball’s number?

(c) What is the probability that the second ball’s number is exactly one greater than the first ball’s number?

(d) Now, assume that after you looked at the first ball, you did not replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

Solution:

(a) Out of $n^2$ pairs of balls that you could have chosen, only one pair $(1, 2)$ corresponds to the event we are interested in, so the probability is $1/n^2$.

(b) The probability that the two balls have the same number is $n/n^2 = 1/n$, so the probability that the balls have different numbers is $1 - 1/n = (n - 1)/n$. By symmetry, it is equally likely for the first ball to have a greater number and for the second ball to have a greater number, so we take the probability $(n - 1)/n$ and divide it by two to obtain $(n - 1)/(2n)$.

(c) Again, there are $n^2$ pairs of balls that we could have drawn, but there are $n - 1$ pairs of balls which correspond to the event we are interested in: $\{(1, 2), (2, 3), \ldots, (n - 1, n)\}$. So, the probability is $(n - 1)/n^2$. 

CS 70, Fall 2019, Discussion 8 3
(d) There are a total of $n(n - 1)$ pairs of balls that we could have drawn, and only the pair $(1, 2)$ corresponds to the event that we are interested in, so the probability is $1/(n(n - 1))$.

The probability that the two balls are the same is now 0, but the symmetry described earlier still applies, so the probability that the second ball has a smaller number is $1/2$.

There are a total of $n(n - 1)$ pairs of balls that we could have drawn, and we are interested in the $n - 1$ pairs $(1, 2), (2, 3), \ldots, (n - 1, n)$ as before. Thus, the probability that the second ball is one greater than the first ball is $1/n$. 