CS 70 Spring 2020	Discrete Mathematics and Probability Theory Course Notes	Dis 08A

1 Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS 70, you are curious to play around with these numbers. Find the probability that:

(a) A given day is both windy and rainy.

(b) A given day is rainy.

(c) For a given pair of days, exactly one of the two days is rainy. (You may assume that the weather on the first day does not affect the weather on the second.)

Solution:

(a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\mathbb{P}(R \mid W) = 0.3$, $\mathbb{P}(R \mid \overline{W}) = 0.8$ and $\mathbb{P}(W) = 0.2$. Then probability that a given day is both rainy and windy is $\mathbb{P}(R \cap W) = \mathbb{P}(R \mid W)\mathbb{P}(W) = 0.3 \times 0.2 = 0.06$.

(b) Probability that it rains on a given day is $\mathbb{P}(R) = \mathbb{P}(R \mid W)\mathbb{P}(W) + \mathbb{P}(R \mid \overline{W})\mathbb{P}(\overline{W}) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$.

(c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the weather on the first day doesn't affect that of the second, $\mathbb{P}(R_1) = \mathbb{P}(R_2) = \mathbb{P}(R)$. The required probability is then just $\mathbb{P}(R_1 \cap \overline{R_2}) + \mathbb{P}(\overline{R_1} \cap R_2) = \mathbb{P}(R_1)\mathbb{P}(\overline{R_2}) + \mathbb{P}(\overline{R_1})\mathbb{P}(R_2) = 2 \cdot 0.7 \cdot 0.3 = 0.42$. Since the weather on the first day does not affect the weather on the second day we can multiply the probabilities.

2 Lie Detector

A lie detector is known to be $4/5$ reliable when the person is guilty and $9/10$ reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only $1/100$ have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is guilty?

Solution:

Let A denote the event that the test indicates that the person is guilty, and B the event that the person is actually guilty. Note that

$$\mathbb{P}[B] = \frac{1}{100}, \quad \mathbb{P}[\overline{B}] = \frac{99}{100}, \quad \mathbb{P}[A \mid B] = \frac{4}{5}, \quad \mathbb{P}[A \mid \overline{B}] = \frac{1}{10}.$$

By Bayes' Rule and the Total Probability Rule the desired probability is

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[B]\mathbb{P}[A \mid B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[B]\mathbb{P}[A \mid B]}{\mathbb{P}[B]\mathbb{P}[A \mid B] + \mathbb{P}[\overline{B}]\mathbb{P}[A \mid \overline{B}]} = \frac{(1/100)(4/5)}{(1/100)(4/5) + (99/100)(1/10)} = \frac{8}{107}$$

3 Bag of Coins

Your friend Forrest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forrest picks a fair coin, and let B be the event that Forrest picks a biased coin. Forrest draws three coins from the bag, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

(b) What is the probability that the third coin he draws is biased?

(c) What is the probability of picking at least two fair coins?

(d) Given that Forrest flips the second coin and sees heads, what is the probability that this coin is biased?

Solution:

(a) The probability of picking F for the first coin is $(n-k)/n$. The probability of picking F for the second coin, after picking one fair coin already is $(n-k-1)/(n-1)$. The probability of picking B for the third coin is $k/(n-2)$. Thus, the probability of picking the exact sequence FFB is

$$\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

(b) One approach is to condition on the possible outcomes for the first and second coins

$$\{FF, FB, BF, BB\}$$

such that

$$\mathbb{P}(T) = \mathbb{P}(T \cap FF) + \mathbb{P}(T \cap FB) + \mathbb{P}(T \cap BF) + \mathbb{P}(T \cap BB)$$

where T is the event that the third coin is biased.

A simpler approach is to use the notion of symmetry. We can envision this by laying out all the coins in a line. If you pick the third coin in the line, the probability that coin is biased is the same as the probability that the first coin in the line is biased, or second, or tenth, etc. If we have no information about any other coins, the probability that any single coin is biased is still k/n.

(c) Note that the probability of picking any sequence of two fair coins and a biased coin is the same. It is in fact the probability from part (a). We need to multiply by the number of arrangements of biased and fair coins, however. So, the probability of picking any sequence with two fair coins is

$$\binom{3}{1}\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

We additionally need to consider the probability of getting 3 fair coins.

$$\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

We simply sum the two to get our answer:

$$\binom{3}{1}\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}+\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

(d) We can apply Bayes Rule. Let H denote the event that Forrest sees heads.

$$\mathbb{P}(B \mid H)=\frac{\mathbb{P}(H \mid B)\mathbb{P}(B)}{\mathbb{P}(H)}$$

Note that $\mathbb{P}(H \mid B)=p$ and that $\mathbb{P}(B)=k/n$. We can now compute the denominator. Using the law of total probability, we can expand $\mathbb{P}(H)$.

$$\begin{aligned}\mathbb{P}(H) &= \mathbb{P}(H \mid B)\mathbb{P}(B)+\mathbb{P}(H \mid F)\mathbb{P}(F)\\ &= p\frac{k}{n}+\frac{1}{2}\frac{n-k}{n}\\ &= \frac{2pk+n-k}{2n}\end{aligned}$$

We now combine both parts to get our answer:

$$\frac{p\cdot(k/n)}{(2pk+n-k)/(2n)}=\frac{2pk}{2pk+n-k}.$$