1. [True or False]

(a) For any three events $A, B, C$, if $\mathbb{P}[A \cap B] > 0$ and $\mathbb{P}[B \cap C] > 0$, then $\mathbb{P}[A \cap C] > 0$.

(b) For events $A, B$ in a uniform probability space, the probability that neither of the events happen is $1 - \mathbb{P}[A] - \mathbb{P}[B]$.

(c) For three events $A, B, C$ in a uniform probability space, the probability that exactly one of the events happens is $\mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - 2\mathbb{P}[A \cap B] - 2\mathbb{P}[A \cap C] - 2\mathbb{P}[B \cap C] + 2\mathbb{P}[A \cap B \cap C]$.

Solution:

(a) False. $A \cap B$ and $B \cap C$ can be non-empty, while $A \cap C = \emptyset$. Take for example $\Omega = \{1, 2, 3, 4\}$ with $\mathbb{P}$ the uniform probability function, and $A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}$, then $\mathbb{P}[A \cap B] = \mathbb{P}\{2\} = 1/4 = \mathbb{P}\{3\} = \mathbb{P}[B \cap C] > 0$, but $\mathbb{P}[A \cap C] = \mathbb{P}[\emptyset] = 0$.

(b) False. The correct expression is $1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A \cap B]$ by inclusion-exclusion.

(c) False. The correct expression is $\mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] - 2\mathbb{P}[A \cap B] - 2\mathbb{P}[A \cap C] - 2\mathbb{P}[B \cap C] + 3\mathbb{P}[A \cap B \cap C]$.

2. [Counting & Probability]

Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 70$, where each $x_i$ is a non-negative integer. We choose one of these solutions uniformly at random.

(a) What is the size of the sample space?

(b) What is the probability that both $x_1 \geq 30$ and $x_2 \geq 30$?

(c) What is the probability that either $x_1 \geq 30$ or $x_2 \geq 30$?

Solution:

(a) $\binom{75}{5}$. This is stars and bars.

(b) Put 30 balls each into the $x_1$ bin and the $x_2$ bin. We are left with 10 balls to distribute, whence there are $\binom{15}{5}$ possibilities. So the probability is $\binom{15}{5}/\binom{75}{5}$.

(c) Let $A_i$ be the event that $x_i \geq 30$, then by inclusion-exclusion $\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \left[\binom{15}{5} + \binom{15}{5} - \binom{15}{5}\right]/\binom{75}{5}$.
3. **[Combinatorial Proof]**

Give a combinatorial proof that \( \binom{n+k-1}{k-1} = \sum_{j=0}^{n} \binom{n-j+k-2}{k-2} \).

**Solution:**

The left-hand side counts the number of ways to throw \( n \) indistinguishable balls into \( k \) distinguishable bins (i.e. stars & bars). The right-hand side is doing the same, by adding up the number of possibilities to throw \( n \) indistinguishable balls into \( k \) bins *where the first bin has exactly \( j \) balls*. Once we place \( j \) balls into the first bin, there are exactly \( n-j \) balls left to distribute over the remaining \( k-1 \) bins, of which there are \( \binom{n-j+k-2}{k-2} \) possibilities.