1 Head Count

Consider a coin with \( P(\text{Heads}) = \frac{2}{5} \). Suppose you flip the coin 20 times, and define \( X \) to be the number of heads.

(a) Name the distribution of \( X \) and what its parameters are.

(b) What is \( P(X = 7) \)?

(c) What is \( P(X \geq 1) \)? Hint: You should be able to do this without a summation.

(d) What is \( P(12 \leq X \leq 14) \)?

Solution:

(a) Since we have 20 independent trials, with each trial having a probability \( \frac{2}{5} \) of success, \( X \sim \text{Binomial}(20, \frac{2}{5}) \).

(b)

\[
P(X = 7) = \binom{20}{7} \left( \frac{2}{5} \right)^7 \left( \frac{3}{5} \right)^{13}.
\]

(c)

\[
P(X \geq 1) = 1 - P(X = 0) = 1 - \left( \frac{3}{5} \right)^{20}.
\]

(d)

\[
P(12 \leq X \leq 14) = P(X = 12) + P(X = 13) + P(X = 14) = \binom{20}{12} \left( \frac{2}{5} \right)^{12} \left( \frac{3}{5} \right)^8 + \binom{20}{13} \left( \frac{2}{5} \right)^{13} \left( \frac{3}{5} \right)^7 + \binom{20}{14} \left( \frac{2}{5} \right)^{14} \left( \frac{3}{5} \right)^6.
\]

2 Exploring the Geometric Distribution

Suppose \( X \sim \text{Geometric}(p) \) and \( Y \sim \text{Geometric}(q) \) are independent. Find the distribution of \( \min\{X,Y\} \) and justify your answer.
Solution:

$x$ is the number of coins we flip until we see a heads from flipping a coin with bias $p$, and $y$ is the same as flipping a coin with bias $q$. Imagine we flip the bias $p$ coin and the bias $q$ coin at the same time. The min of the two random variables represents how many simultaneous flips occur before at least one head is seen.

The probability of not seeing a head at all on any given simultaneous flip is $(1 - p)(1 - q)$, so the probability that there will be a success on any particular trial is $p + q - pq$. Therefore, $\min\{x, y\} \sim \text{Geometric}(p + q - pq)$.

We can also solve it algebraically. The probability that $\min\{x, y\} = k$ for some positive integer $k$ is the probability that the first $k-1$ coin flips for both $x$ and $y$ were tails, then times the probability that we get heads on the $k$-th toss. Specifically,

$$(1 - p)(1 - q)^{k-1} \cdot (p + q - pq)$$

We recognize this as the formula for a geometric random variable with parameter $p + q - pq$.

3 The Memoryless Property

Let $x$ be a discrete random variable which takes on values in $\mathbb{Z}_+$. Suppose that for all $m, n \in \mathbb{N}$, we have $\Pr(x > m + n | x > n) = \Pr(x > m)$. Prove that $x$ is a geometric distribution. Hint: In order to prove that $x$ is geometric, it suffices to prove that there exists a $p \in [0, 1]$ such that $\Pr(x > i) = (1 - p)^i$ for all $i > 0$.

Solution:

Notice that

$$\Pr(x > m + n | x > n) = \frac{\Pr(x > m + n)}{\Pr(x > n)} = \Pr(x > m),$$

where the first equality holds from definition of conditional probability, and the second from the given in the question. So, this gives $\Pr(x > m + n) = \Pr(x > m)\Pr(x > n)$.

$$\Pr(x > m) = \Pr(x > m + n | x > n) = \frac{\Pr(x > m + n)}{\Pr(x > n)},$$

where that the first equality comes from the given in the question, and the second equality holds from definition of conditional. So, this gives $\Pr(x > m + n) = \Pr(x > m)\Pr(x > n)$.

By repeatedly applying this property, we can deduce $\Pr(x > n) = \Pr(x > 1 + \cdots + 1) = \Pr(x > 1)^n$. Let $p := 1 - \Pr(x > 1)$. We see that $\Pr(x > n) = (1 - p)^n$, which is the tail probability of the geometric distribution, and hence $x \sim \text{Geo}(p)$.

4 Cookie Jars

You have two jars of cookies, each of which starts with $n$ cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$) and
eat one cookie from that jar. One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let $X$ be the random variable representing the number of remaining cookies in non-empty jar at that time. What is the distribution of $X$?

**Solution:** Assume that you found jar 1 empty. The probability that $X = k$ and you found jar 1 empty is computed as follows. In order for there to be $k$ cookies remaining, you must have eaten a cookie for $2n - k$ days, and then you must have chosen jar 1 (to discover that it is empty). Within those $2n - k$ days, exactly $n$ of those days you chose jar 1. The probability of this is $\binom{2n-k}{n} 2^{-(2n-k)}$. Furthermore, the probability that you then discover jar 1 is empty the day after is $1/2$. So, the probability that $X = k$ and you discover jar 1 empty is $\binom{2n-k}{n} 2^{-(2n-k+1)}$. However, we assumed that we discovered jar 1 to be empty; the probability that $X = k$ and jar 2 is empty is the same by symmetry, so the overall probability that $X = k$ is:

$$
P(X = k) = \binom{2n-k}{n} \frac{1}{2^{2n-k}}, \quad k \in \{0, \ldots, n\}.
$$