1. [True or False]

(a) If \( P(A) > 0 \) and \( P(B) > 0 \), and \( A \) and \( B \) are disjoint, then \( A \) and \( B \) are not independent.

(b) If for three events \( A, B \) and \( C \), it is true that \( P(A \mid C) > P(B \mid C) \), then \( P(A) > P(B) \).

(c) For independent events \( A \) and \( B \), \( P(A \cup B) = P(A) + P(B) \).

**Solution:**

(a) **True.** \( P(A \mid B) = 0 \neq P(A) \).

(b) **False.** Throw for example a fair five-sided die, and let \( A = \{2, 4\}, B = \{1, 3, 5\} \) and \( C = \{2, 3, 4\} \).

(c) **False.** The correct expression is \( P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B) \). Any two non-empty independent events provide a counterexample to the claim in the problem statement.

2. [Short Answer] Consider the four ace cards (remember, the suits are hearts and diamond, which are red, and spades and clubs, which are black). Melissa shuffles these four cards and draws the top two cards.

(a) Let \( A \) be the event that Melissa has the ace of hearts. Given \( A \), what is the probability that Melissa has both red cards?

(b) Given that Melissa has at least one red card, what is the probability that she has both red cards?

(c) Give an event \( B \) that is independent of \( A \).

**Solution:**

(a) \( \frac{1}{5} \). Each of the six hands is equally likely. Three of them contain the ace of hearts, and of those one contains both red cards. Therefore \( P(\text{both red cards} \mid \text{ace of hearts}) = \frac{1}{3} \).

(b) \( \frac{1}{5} \). There are five hands that contain at least one red card (since only one contains both black cards). Of those one contains both red cards. Therefore \( P(\text{both red cards} \mid \text{at least one red card}) = \frac{1}{5} \).

(c) One example is \( B = \{\text{Melissa has the ace of spades and a red card}\} \): \( P(A \cap B) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P(A) \cdot P(B) \).
3. [Long Answer] You select a three digit decimal uniformly in \{000, 001, \ldots, 999\} (note that we consider, e.g., 023 to be a three digit decimal number). What is the probability that the number has three identical digits given that it has at least two identical digits?

Solution:

We first count the number of three digit numbers that have exactly two identical digits: There are 3 ways of choosing the location of the two digits that are identical, and then there are 10 ways to choose what the identical digits are, and 9 choices for what the remaining digit is. That is, we have a total of \(3 \cdot 9 \cdot 10 = 270\) three digit numbers with two identical digits. Moreover, there are exactly 10 three digit numbers all of whose digits are identical, and so the probability in question is \(\frac{10}{270+10} = \frac{1}{28}\).