1 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) Name the distribution of X and what its parameters are.

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

Solution:

(a) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(b)

$$\mathbb{P}(X = 7) = \binom{20}{7}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right)^{13}.$$

(c)

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}(12 \leq X \leq 14) &= \mathbb{P}(X = 12) + \mathbb{P}(X = 13) + \mathbb{P}(X = 14) \\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$

2 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C=1$	$C=2$	$C=3$
$G=0$			
$G=1$			

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}(G=0)$	
$\mathbb{P}(G=1)$	

$\mathbb{P}(C=1)$	$\mathbb{P}(C=2)$	$\mathbb{P}(C=3)$

(d) Are G and C independent?

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

Solution:

(a) The sample space is the set of all possible sequences of children that the Browns can have: $\Omega = \{g, bg, bbg, bbb\}$. The probabilities of these sample points are:

$$\mathbb{P}(g) = \frac{1}{2}$$
$$\mathbb{P}(bg) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$
$$\mathbb{P}(bbg) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$\mathbb{P}(bbb) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

(b)

	$C=1$	$C=2$	$C=3$
$G=0$	0	0	$\mathbb{P}(bbb) = 1/8$
$G=1$	$\mathbb{P}(g) = 1/2$	$\mathbb{P}(bg) = 1/4$	$\mathbb{P}(bbg) = 1/8$

(c) Marginal distribution for G:

$$\mathbb{P}(G=0) = 0 + 0 + \frac{1}{8} = \frac{1}{8}$$
$$\mathbb{P}(G=1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Marginal distribution for C:

$$\mathbb{P}(C=1) = 0 + \frac{1}{2} = \frac{1}{2}$$
$$\mathbb{P}(C=2) = 0 + \frac{1}{4} = \frac{1}{4}$$
$$\mathbb{P}(C=3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

(d) No, G and C are not independent. If two random variables are independent, then

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y).$$

To show this dependence, consider an entry in the joint distribution table, such as $\mathbb{P}(G=0, C=3) = 1/8$. This is not equal to $\mathbb{P}(G=0)\mathbb{P}(C=3) = (1/8)\cdot(1/4) = 1/32$, so the random variables are not independent.

(e) We can apply the definition of expectation directly for this problem, since we've computed the marginal distribution for both random variables.

$$\mathbb{E}(G) = 0\cdot\mathbb{P}(G=0) + 1\cdot\mathbb{P}(G=1) = 1\cdot\frac{7}{8} = \frac{7}{8}$$
$$\mathbb{E}(C) = 1\cdot\mathbb{P}(C=1) + 2\cdot\mathbb{P}(C=2) + 3\cdot\mathbb{P}(C=3) = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4} = \frac{7}{4}$$

3 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$, $\mathbb{P}(X=1)$, $\mathbb{P}(X=2)$ and $\mathbb{P}(X=3)$?

(b) What do your answers you computed in part a add up to?

(c) Compute $\mathbb{E}(X)$ from the definition of expectation.

(d) Let X_i be an indicator random variable that equals 1 if the ith card a is queen and 0 otherwise. Are the X_i indicators independent?

Solution:

(a) Calculate each case of X = 0, 1, 2, 3:

We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X=0) = \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X=0)=\frac{\binom{48}{3}}{\binom{52}{3}}=\frac{4324}{5525}$$

- We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X=1)=\frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}}=\frac{1128}{5525}$$

- Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X=2)=\frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}}=\frac{72}{5525}$$

- Choose 3 queens out of 4.

$$\mathbb{P}(X=3)=\frac{\binom{4}{3}}{\binom{52}{3}}=\frac{1}{5525}$$

(b) We check:

$$\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3)=\frac{4324+1128+72+1}{5525}=1$$

(c) From the definition, $\mathbb{E}(X)=\sum_{k=0}^{3}k\mathbb{P}(X=k)$, so

$$\mathbb{E}(X)=0\cdot\frac{4324}{5525}+1\cdot\frac{1128}{5525}+2\cdot\frac{72}{5525}+3\cdot\frac{1}{5525}=\frac{3}{13}.$$

(d) No, they are not independent. As an example:

$$\mathbb{P}(X_1=1)\mathbb{P}(X_2=1)=\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$$

However,

$$\mathbb{P}(X_1=1,X_2=1)=\mathbb{P}(\text{the first and second cards are both queens})=\frac{4}{52}\cdot\frac{3}{51}=\frac{1}{221}.$$