CS 70 Discrete Mathematics and Probability Theory
Spring 2020

HW 0

1 Homework Process and Study Group

Responding to this set of questions is required.

2 Administrivia

(a) Make sure you are on the course Piazza (for Q&A) and Gradescope (for submitting homeworks, including this one). Find and familiarize yourself with the course website. What is its homepage's URL?

(b) Read the policies page on the course website. What is the percentage breakdown of how your grade is calculated?

Solution:

(a) The course website is located at `http://www.eecs70.org/`.

(b) Homework is worth 13.33% of your grade, each midterm is worth 21.66%, and the final is worth 43.33%.

3 Course Policies

Go to the course website and read the course policies carefully. Leave a followup in the Homework 0, Question 2 thread on Piazza if you have any questions. Are the following situations violations of course policy? Write "Yes" or "No", and a short explanation for each.

(a) Alice and Bob work on a problem in a study group. They write up a solution together and submit it, noting on their submissions that they wrote up their homework answers together.

(b) Carol goes to a homework party and listens to Dan describe his approach to a problem on the board, taking notes in the process. She writes up her homework submission from her notes, crediting Dan.

(c) Erin finds a solution to a homework problem on a website. She reads it and then, after she has understood it, writes her own solution using the same approach. She submits the homework with a citation to the website.

(d) Frank is having trouble with his homework and asks Grace for help. Grace lets Frank look at her written solution. Frank copies it onto his notebook and uses the copy to write and submit his homework, crediting Grace.

(e) Heidi has completed her homework using LaTeX. Her friend Irene has been working on a homework problem for hours, and asks Heidi for help. Heidi sends Irene her PDF solution, and Irene uses it to write her own solution with a citation to Heidi.

Solution:

(a) Yes, this is a violation of course policy. All solutions must be written entirely by the student submitting the homework. Even if students collaborate, each student must write a unique, individual solution. In this case, both Alice and Bob would be culpable.

(b) No, this is not a violation of course policy. While sharing *written solutions* is not allowed, sharing *approaches* to problems is allowed and encouraged. Because Carol only copied down *notes*, not *Dan's solution*, and properly cited Dan's contribution, this is an actively encouraged form of collaboration.

(c) No, this is not a violation of course policy. Using external sources to help with homework problems, while less encouraged than peer collaboration, is fine as long as (i) the student makes sure to understand the solution; (ii) the student uses understanding to write a new solution, and does not copy from the external source; and (iii) the student credits the external source.

(d) Yes, this is a violation of course policy, and both Frank and Grace would be culpable. Even though Frank credits Grace, written solutions should never be shared in the first place, and certainly not copied down. This is to ensure that each student learns how to write and present clear and convincing arguments. To be safe, try not to let anybody see your written solutions at any point in the course—restrict your collaboration to *approaches* and *verbal communication*.

(e) Yes, this is a violation of course policy. Once again, a citation does not make up for the fact that written solutions should never be shared, in written or typed form. In this case, both Heidi and Irene would be culpable.

4 Use of Piazza

Piazza is incredibly useful for Q&A in such a large-scale class. We will use Piazza for all important announcements. You should check it frequently. We also highly encourage you to use Piazza to ask questions and answer questions from your fellow students.

(a) Navigate to the "Index" Piazza post, where you can find links to most resources in the course. Write down the Piazza post number for the Note 0 Thread. (When you see @x on Piazza, where x is a positive integer, then x is the post number of the linked post.)

(b) Read the Piazza Etiquette section of the course policies and explain what is wrong with the following hypothetical student question: "Can someone explain the proof of Theorem XYZ to me?" (Assume Theorem XYZ is a complicated concept.)

(c) When are the weekly posts released? Are they required reading?

Solution:

(a) The post number for the Note 0 Thread is 55. 15 (for Note 1) is also acceptable, since Note 0 was not posted for some time.

(b) There are two things wrong with this question. First, this question does not pass the 5-minute test. This concept takes longer than 5 minutes to explain, and therefore is better suited to Office Hours. Second, this question does not hone in on a particular concept with which the student is struggling. Questions on Piazza should be narrow, and should include every step of reasoning that led up to the question. A better question in this case might be: "I understood every step of the proof of Theorem XYZ in Note 2, except for the very last step. I tried to reason it like this, but I didn't see how it yielded the result. Can someone explain where I went wrong?"

(c) The weekly posts are released every Sunday. They're required reading.

5 LaTeX

If you have ample time on your hands and would like to learn a new skill, you may be interested in learning LaTeX! LaTeX is a document preparation system that puts mathematical formulae into nicely formatted documents. Using LaTeX can help you organize your thought process and make lives easier for readers. We have provided some resources on the course website to help you get started with using LaTeX. Feel free to ask questions on Piazza if you have any questions.

For this question, try to typeset the following formulas. This will give you some practice writing mathematical formulas properly. If you choose hand-write and scan your solutions, just write out the formulas by hand.

Hint: You may find the *amsmath* LaTeX package helpful.

(a) $\forall x \exists y \left(\left(P(x) \wedge Q(x, y)\right) \implies x \le \sqrt{y}\right)$

(b) $\displaystyle\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

Solution:

(a)
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$\forall x \exists y
\left(\left(P(x) \wedge Q(x, y)\right) \implies
x \le \sqrt{y}\right)$
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(b) `$\displaystyle \sum_{i = 0}^k i = \frac{k(k + 1)}{2}$`

6 Set Operations

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$, the set of integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{N}$, the set of natural numbers: $\{0, 1, 2, 3, \ldots\}$

(a) Given a set $A = \{1, 2, 3, 4\}$, what is $\mathscr{P}(A)$ (Power Set)?

(b) Given a generic set B, how do you describe $\mathscr{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

(c) What is $\mathbb{R} \cap \mathscr{P}(A)$?

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

(f) What kind of numbers are in $\mathbb{R} \setminus \mathbb{Q}$?

(g) If $S \subseteq T$, what is $S \setminus T$?

Solution:

(a)

$$\mathscr{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$

(b) $\mathscr{P}(B) = \{T \mid T \subseteq B\}$

(c) $\{\}$ or $\varnothing$

(d) $\mathbb{Z}$

(e) $\mathbb{Q}$

(f) The set of irrational numbers

(g) $\varnothing$

7 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(d) $f(A \cup B) = f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(f) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x) \in A \cup B$. Then either $f(x) \in A$, in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$. This proves that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(c) Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(d) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

(e) Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(f) Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.

8 Linear Algebra

Let $A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -2 & 3 & -3 & -1 \\ 3 & -3 & 6 & 7 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$.

(a) Do the columns of A span $\mathbb{R}^3$? Justify your answer.

(b) Are the columns of A linearly independent? Justify your answer.

(c) Describe all solutions to $Ax = b$ in parametric vector form. Do the same for $Ax = 0$.

(d) Provide an example of each of the following:

   (i) An invertible matrix

   (ii) A matrix with linearly independent columns, but linearly dependent rows

   (iii) A 2x2 matrix with two real eigenvalues that are different from each other

   (iv) A 2x2 matrix with a single real eigenvalue and only a single eigenvector

   (v) A system of 2 equations in 2 unknowns with no solutions

(e) A new streaming service charges 5 dollars per month for students, and 10 dollars per month for everyone else. This month, the service had 55 users, and collected 425 dollars. Set up a system of linear equations, and find the number of students using the service this month.

Solution:

(a) We inspect the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It contains a pivot in every row, so the columns span $\mathbb{R}^3$.

(b) Not every column has a pivot, so the columns of A are not linearly independent.

(c) Reduce the augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 2 & 1 \\ -2 & 3 & -3 & -1 & -3 \\ 3 & -3 & 6 & 7 & 3 \end{bmatrix}.$$

One possible form is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now that we are in reduced echelon form, we can describe our solutions. Note that x_3 is a free variable. We have the following: $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Following the same procedure for $Ax = 0$, we obtain the solution set given by

$$x_3 \cdot \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

(d) There are multiple correct answers for this question.

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(e) Let s be the number of users that are students, and e be the number of other users.

$$\begin{bmatrix}1 & 1\\5 & 10\end{bmatrix}\begin{bmatrix}s\\e\end{bmatrix} = \begin{bmatrix}55\\425\end{bmatrix}$$

Solving this system, we get $s = 25$ and $e = 30$.

9 Calculus

(a) Compute a closed-form expression for the value of following summation:

$$\sum_{k=1}^{\infty} \frac{9}{2^k}$$

(b) Use summation notion to write an expression equivalent to the following statement:

The sum of the first n consecutive odd integers, starting from 1

(c) Compute the following integral:

$$\int_0^{\infty} \sin(t)e^{-t}dt$$

(d) Find the maximum value of the following function and determine where it occurs:

$$f(x) = -x \cdot \ln x$$

Solution:

(a) Use the convergence of geometric series with $|r| < 1$.

$$\sum_{k=1}^{\infty} \frac{9}{2^k} = 9 \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = 9 \cdot (\sum_{k=0}^{\infty} \frac{1}{2^k} - 1)$$
$$= 9 \cdot (2-1) = 9$$

(b) Observe that $2k+1$ is odd for all $k \in \mathbb{Z}$.

$$\sum_{k=0}^{n-1} 2k+1$$

(c) Let $I = \int sin(t)e^{-t}$.
Use integration by parts, with $u = \sin(t)$ and $dv = e^{-t}$.
This means $du = \cos(t)$ and $v = -e^{-t}$.

$$I = \int \sin(t)e^{-t}dt = uv - \int v \cdot du$$
$$= -\sin(t)e^{-t} + \int e^{-t}\cos(t)dt$$

Use integration by parts again on $\int e^{-t}\cos(t)dt$, with $u = \cos(t)$ and $dv = e^{-t}$. This means $du = -\sin(t)$ and $v = -e^{-t}$.

$$\begin{aligned}\int e^{-t}\cos(t)dt &= uv - \int v \cdot du \\ &= -\cos(t)e^{-t} - \int e^{-t} \cdot \sin(t)dt \\ &= -\cos(t)e^{-t} - I\end{aligned}$$

Combining these results:

$$\begin{aligned} I &= -\sin(t)e^{-t} - \cos(t)e^{-t} - I \\ \Rightarrow 2I &= -\sin(t)e^{-t} - \cos(t)e^{-t} \\ \Rightarrow I &= \frac{-\sin(t)e^{-t} - \cos(t)e^{-t}}{2}\end{aligned}$$

Finally, we have:

$$I\Big|_0^\infty = \frac{0-0}{2} - \frac{0-1}{2} = \frac{1}{2}$$

(d) Compute the derivative of the function, and set it equal to 0.

$$\begin{aligned}\frac{\partial f}{\partial x} &= -1 \cdot \ln x + -x \cdot \frac{1}{x} \\ &= -\ln x - 1 = 0 \\ &\Rightarrow x^* = \frac{1}{e}\end{aligned}$$

The optimal value is achieved at $x^* = \frac{1}{e}$, and the corresponding value is $f(x^*) = \frac{1}{e}$.

10 Academic Integrity

Please write or type out the following pledge in print, and sign it.

> I pledge to uphold the university's honor code: to act with honesty, integrity, and respect for others, including their work. By signing, I ensure that all written homework I submit will be in my own words, that I will acknowledge any collaboration or help received, and that I will neither give nor receive help on any examinations.