CS 70 Discrete Mathematics and Probability Theory
Fall 2017 Satish Rao and Kannan Ramchandran

HW 1

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 Always True or Always False?

Classify the following statements as being one of the following and justify your answers.

- True for all combinations of x and y (Tautology)
- False for all combinations of x and y (Contradiction)
- Neither

(a) $x \wedge (x \implies y) \wedge (\neg y)$

(b) $x \implies (x \vee y)$

(c) $(x \vee y) \vee (x \vee \neg y)$

(d) $(x \implies y) \vee (x \implies \neg y)$

(e) $(x \vee y) \wedge (\neg(x \wedge y))$

(f) $(x \implies y) \wedge (\neg x \implies y) \wedge (\neg y)$

Solution:

(a) **Contradiction**

x	y	$x \implies y$	$x \wedge (x \implies y) \wedge (\neg y)$
T	T	T	F
T	F	F	F
F	T	T	F
F	F	T	F

(b) **Tautology**

x	y	$x \vee y$	$x \implies (x \vee y)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(c) **Tautology**

x	y	$x \vee y$	$x \vee \neg y$	$(x \vee y) \vee (x \vee \neg y)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(d) **Tautology**

x	y	$x \implies y$	$x \implies \neg y$	$(x \implies y) \vee (x \implies \neg y)$
T	T	T	F	T
T	F	F	T	T
F	T	T	T	T
F	F	T	T	T

(e) **Neither**

x	y	$x \vee y$	$\neg(x \wedge y)$	$(x \vee y) \wedge (\neg(x \wedge y))$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

(f) **Contradiction**

x	y	$x \implies y$	$\neg x \implies y$	$\neg y$	$(x \implies y) \wedge (\neg x \implies y) \wedge (\neg y)$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	F
F	F	T	F	T	F

2 Lewis Carroll

Here is an extract from Lewis Carroll's treatise *Symbolic Logic* of 1896:

(I) No one, who is going to a party, ever fails to brush his or her hair.

(II) No one looks fascinating, if he or she is untidy.

(III) Opium-eaters have no self-command.

(IV) Everyone who has brushed his or her hair looks fascinating.

(V) No one wears kid gloves, unless he or she is going to a party.

(VI) A person is always untidy if he or she has no self-command.

(a) Write each of the above six sentences as a quantified proposition over the universe of all people. You should use the following symbols for the various elementary propositions: $P(x)$ for "x goes to a party", $B(x)$ for "x has brushed his or her hair", $F(x)$ for "x looks fascinating", $U(x)$ for "x is untidy", $O(x)$ for "x is an opium-eater", $N(x)$ for "x has no self-command", and $K(x)$ for "x wears kid gloves".

(b) Now rewrite each proposition equivalently using the contrapositive.

(c) You now have twelve propositions in total. What can you conclude from them about a person who wears kid gloves? Explain clearly the implications you used to arrive at your conclusion.

Solution:

(a) (I) "No one, who is going to a party, ever fails to brush his or her hair"
Answer: $\forall x \, \big(P(x) \Rightarrow B(x)\big)$

(II) "No one looks fascinating, if he or she is untidy."
Answer: $\forall x \, \big(U(x) \Rightarrow \neg F(x)\big)$

(III) "Opium-eaters have no self-command."
Answer: $\forall x \, \big(O(x) \Rightarrow N(x)\big)$

(IV) "Everyone who has brushed his or her hair looks fascinating"
Answer: $\forall x \, \big(B(x) \Rightarrow F(x)\big)$

(V) "No one wears kid gloves, unless he or she is going to a party"
Answer: $\forall x \, \big(K(x) \Rightarrow P(x)\big)$

(VI) "A person is always untidy if he or she has no self-command."
Answer: $\forall x \, \big(N(x) \Rightarrow U(x)\big)$

(b) (I) $\forall x \, \big(\neg B(x) \Rightarrow \neg P(x)\big)$

(II) $\forall x \, \big(F(x) \Rightarrow \neg U(x)\big)$

(III) $\forall x \, \big(\neg N(x) \Rightarrow \neg O(x)\big)$

(IV) $\forall x \, \big(\neg F(x) \Rightarrow \neg B(x)\big)$

(V) $\forall x \, \big(\neg P(x) \Rightarrow \neg K(x)\big)$

(VI) $\forall x \, \big(\neg U(x) \Rightarrow \neg N(x)\big)$

(c) **Answer**: A person who wears kid gloves is not an opium-eater.
Derivation: $K(x) \Rightarrow P(x) \Rightarrow B(x) \Rightarrow F(x) \Rightarrow \neg U(x) \Rightarrow \neg N(x) \Rightarrow \neg O(x)$

3 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$	$\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big)$
(b)	$\neg\exists x\ \forall y\ \big(P(x,y) \Rightarrow \neg Q(x,y)\big)$	$\forall x\ \big((\exists y\ P(x,y)) \land (\exists y\ Q(x,y))\big)$
(c)	$\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big)$	$\forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$

Solution:

(a) Not equivalent.

Justification: We can rewrite the left side as as $\forall x\ \big((\neg(\exists y\ Q(x,y))) \lor P(x)\big)$ and the right side as $\forall x\ \exists y\ \big(\neg Q(x,y) \lor P(x)\big)$ Applying the negation on the left side of the equivalence $(\neg(\exists y Q(x,y)))$ changes the $\exists y$ to $\forall y$, and the two sides are clearly not the same. Another approach to the problem is to consider by linguistic example. Let x and y span the universe of all people, and let $Q(x,y)$ mean "Person x is Person y's offspring", and let $P(x)$ mean "Person x likes tofu". The right side claims that, for all Persons x, there exists some Person y such that either Person x is not Person y's offspring or that Person x likes tofu. The left side claims that, for all Persons x, if there exists a parent of Person x, then Person x likes tofu. Obviously, these are not the same.

(b) Not equivalent.

Justification: Using De Morgan's Law to distribute the negation on the left side yields

$$\forall x\ \exists y\ \big(P(x,y) \land Q(x,y)\big).$$

But $\exists$ does not distribute over $\land$. There could exist different values of y such that $P(x,y)$ and $Q(x,y)$ for a given x, but not necessarily the same value.

(c) Equivalent.

Justification: We can rewrite the left side as $\forall x\ \exists y\ \big(\neg P(x) \lor Q(x,y)\big)$ and the right side as $\forall x\ \big(\neg P(x) \lor (\exists y\ Q(x,y))\big)$. Clearly, the two sides are the same if $\neg P(x)$ is true. If $\neg P(x)$ is false, then the two sides are still the same, because $\forall x\ \exists y\ \big(\text{False} \lor Q(x,y)\big) \equiv \forall x\ \big(\text{False} \lor (\exists y\ Q(x,y))\big)$.

4 Karnaugh Maps

Below is the truth table for the boolean function

$$Y = (\neg A \land \neg B \land C) \lor (\neg A \land B \land \neg C) \lor (A \land \neg B \land C) \lor (A \land B \land C).$$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

In this question, we will explore a different way of representing a truth table, the *Karnaugh map*. A Karnaugh map is just a grid-like representation of a truth table, but as we will see, the mode of presentation can give more insight. The values inside the squares are copied from the output column of the truth table, so there is one square in the map for every row in the truth table.

Around the edge of the Karnaugh map are the values of the input variables. Note that the sequence of numbers across the top of the map is not in binary sequence, which would be 00, 01, 10, 11. It is instead 00, 01, 11, 10, which is called *Gray code* sequence. Gray code sequence only changes one binary bit as we go from one number to the next in the sequence. That means that adjacent cells will only vary by one bit, or Boolean variable. In other words, *cells sharing common Boolean variables are adjacent*.

For example, here is the Karnaugh map for Y:

A \ BC	00	01	11	10
0	0	1	0	1
1	0	1	1	0

The Karnaugh map provides a simple and straight-forward method of minimizing boolean expressions by visual inspection. The technique is to examine the Karnaugh map for any groups of adjacent ones that occur, which can be combined to simplify the expression. Note that "adjacent" here means in the modular sense, so adjacency wraps around the top/bottom and left/right of the Karnaugh map; for example, the top-most cell of a column is adjacent to the bottom-most cell of the column.

For example, the ones in the second column in the Karnaugh map above can be combined because $(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C)$ simplifies to $(\neg B \wedge C)$. Applying this technique to the Karnaugh map (illustrated below), we obtain the following simplified expression for Y:

$$Y = (\neg B \wedge C) \vee (A \wedge C) \vee (\neg A \wedge B \wedge \neg C).$$

A \ BC	00	01	11	10
0	0	1	0	1
1	0	1	1	0

(a) Write the truth table for the boolean function

$$Z = (\neg A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (\neg A \wedge \neg B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \\ \vee (A \wedge \neg B \wedge C \wedge \neg D).$$

(b) Using your truth table, fill in the Karnaugh map for Z below.

AB \ CD	00	01	11	10
00				
01				
11				
10				

(c) Using your Karnaugh map, write down a simplified expression for Z.

(d) Show that this simplification could also be found algebraically by factoring the expression for Z.

Solution:

(a) Here is the truth table:

A	B	C	D	Z
T	T	T	T	F
T	T	T	F	F
T	T	F	T	F
T	T	F	F	F
T	F	T	T	F
T	F	T	F	T
T	F	F	T	F
T	F	F	F	T
F	T	T	T	F
F	T	T	F	F
F	T	F	T	F
F	T	F	F	F
F	F	T	T	F
F	F	T	F	T
F	F	F	T	F
F	F	F	F	T

CD

		00	01	11	10
	00				
AB	01				
	11				
	10				

(b) The four corners should contain 1, the rest of the entries are 0.

(c) The four corners can be combined to get

$$Z = \neg B \wedge \neg D.$$

(d) Using the distributive property, the terms can be simplified:

$$\begin{aligned} Z &= ((\neg A \wedge \neg B \wedge \neg D) \wedge (\neg C \vee C)) \vee ((A \wedge \neg B \wedge \neg D) \wedge (\neg C \vee C)) \\ &= ((\neg A \wedge \neg B \wedge \neg D) \wedge \mathbf{T}) \vee ((A \wedge \neg B \wedge \neg D) \wedge \mathbf{T}) \\ &= (\neg A \wedge \neg B \wedge \neg D) \vee (A \wedge \neg B \wedge \neg D) = (\neg B \wedge \neg D) \wedge (\neg A \vee A) = (\neg B \wedge \neg D) \wedge \mathbf{T} \\ &= \neg B \wedge \neg D \end{aligned}$$

5 Social Network

Suppose that $p_1, p_2, \ldots, p_n$ denote n people where every two people are either friends or strangers. Let Friends(x,y) be the predicate "x and y are friends". Prove or provide a counterexample for the following statements.

(a) For all cases with $n = 5$ people, there exists a group of 3 people that are either all friends or all strangers. In mathematical notation we write this as:

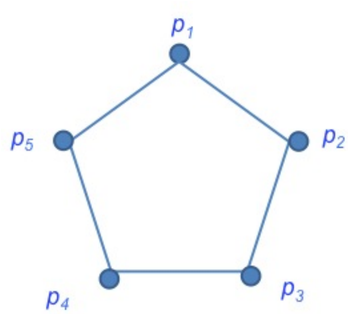
$$\exists (i,j,k) \in \{1,2,\ldots,5\}^3 \text{ such that } i < j < k \text{ and } (\text{Friends}(p_i,p_j) \wedge \text{Friends}(p_j,p_k) \wedge \text{Friends}(p_i,p_k)) \vee (\neg\text{Friends}(p_i,p_j) \wedge \neg\text{Friends}(p_j,p_k) \wedge \neg\text{Friends}(p_i,p_k)).$$

(b) For all cases with $n = 6$ people, there exists a group of 3 people that are either all friends or all strangers. In mathematical notation we write this as:

$$\exists (i,j,k) \in \{1,2,\ldots,6\}^3 \text{ such that } i < j < k \text{ and } (\text{Friends}(p_i,p_j) \wedge \text{Friends}(p_j,p_k) \wedge \text{Friends}(p_i,p_k)) \vee (\neg\text{Friends}(p_i,p_j) \wedge \neg\text{Friends}(p_j,p_k) \wedge \neg\text{Friends}(p_i,p_k)).$$

Solution:

(a) The statement is false. A counterexample is shown below where people are connected if they are friends and unconnected if they are strangers. In this example, at most 2 are friends or strangers.



(b) The statement is true.

Proof: For any person p_i, we could divide the rest of people into 2 groups: the group of p_i's friends and the group of strangers. One of the groups must have at least 3 people.

Case (1): At least 3 people are p_i's friends.

Let $\{j,k,l\} \subset \{1,2,\ldots,6\} \setminus \{i\}$ and $j < k < l$ (distinct) such that

$$(\text{Friends}(p_j,p_i) \wedge \text{Friends}(p_k,p_i) \wedge \text{Friends}(p_l,p_i)).$$

If p_j, p_k, and p_l are all strangers, the statement is true. If not, $\exists\{m,n\} \subset \{i,j,k\}$ where $m < n$ (distinct) such that Friends(p_m,p_n). Then p_m, p_n, and p_i are all friends.

Case (2): At least 3 people are strangers to p_i.

Let $\{j,k,l\} \subset \{1,2,\ldots,6\} \setminus \{i\}$ and $j < k < l$ (distinct) such that

$$(\neg\text{Friends}(p_j, p_i) \wedge \neg\text{Friends}(p_k, p_i) \wedge \neg\text{Friends}(p_l, p_i)).$$

If p_j, p_k, and p_l are all friends, the statement is true. If not, $\exists\{m,n\} \subset \{i,j,k\}$ where $m < n$ (distinct) such that $\neg\text{Friends}(p_m, p_n)$. Then p_m, p_n, and p_i are all strangers.

6 Prove or Disprove

(a) $\forall n \in \mathbb{N}$, if n is odd then $n^2 + 2n$ is odd.

(b) $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x + y - |x - y|)/2$.

(c) $\forall a, b \in \mathbb{R}$ if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.

(d) $\forall r \in \mathbb{R}$, if r is irrational then $r + 1$ is irrational.

(e) $\forall n \in \mathbb{N}$, $10n^2 > n!$.

Solution:

(a) **Answer**: True.
Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 2n$, we get $(2k + 1)^2 + 2 \times (2k + 1)$. Simplifying the expression yields $4k^2 + 8k + 3$. This can be rewritten as $2 \times (2k^2 + 4k + 1) + 1$. Since $2k^2 + 4k + 1$ is a natural number, by the definition of odd numbers, $n^2 + 2n$ is odd.
Alternatively, we could also factor the expression to get $n(n + 2)$. Since n is odd, $n + 2$ is also odd. The product of 2 odd numbers is also an odd number. Hence $n^2 + 2n$ is odd.

(b) **Answer**: True.
Proof: We will use a proof by cases. We know the following about the absolute value function for real number z.

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x - y| = y - x$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - y + x}{2} = x = \min(x, y).$$

Case 2: $x \geq y$. This means $|x - y| = x - y$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - x + y}{2} = y = \min(x, y).$$

(c) **Answer**: True.
Proof: We will use a proof by contraposition. Suppose that $a > 7$ and $b > 3$ (note that this is equivalent to $\neg\big(a \le 7 \lor b \le 3\big)$). Since $a > 7$ and $b > 3$, $a+b > 10$ (note that $a+b > 10$ is equivalent to $\neg\big(a+b \le 10\big)$). Thus, if $a+b \le 10$, then $a \le 7$ or $b \le 3$ (or both, as "or" is not "exclusive or" in this case).

(d) **Answer**: True.
Proof: We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form a/b where a and b are integers. Then r can be written as $(a-b)/b$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers. By contraposition, if r is irrational, then $r+1$ is irrational.

(e) **Answer**: False.
Proof: We will use proof by counterexample. Let $n = 6$. $10 \times 6^2 = 360$. $6! = 720$. Since $10n^2 < n!$, the claim is false.