CS 70 Discrete Mathematics and Probability Theory
Summer 2018 Sinho Chewi and Vrettos Moulos **HW 1**

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. (*signature here*)

1 Always True or Always False?

Classify the following statements as being one of the following and justify your answers.

- True for all combinations of x and y (Tautology)
- False for all combinations of x and y (Contradiction)
- Neither

(a) $x \wedge (x \implies y) \wedge (\neg y)$

(b) $x \implies (x \vee y)$

(c) $(x \vee y) \vee (x \vee \neg y)$

(d) $(x \implies y) \vee (x \implies \neg y)$

(e) $(x \vee y) \wedge (\neg(x \wedge y))$

(f) $(x \implies y) \wedge (\neg x \implies y) \wedge (\neg y)$

Solution:

(a) **Contradiction**

x	y	$x \implies y$	$x \wedge (x \implies y) \wedge (\neg y)$
T	T	T	F
T	F	F	F
F	T	T	F
F	F	T	F

(b) **Tautology**

x	y	$x \vee y$	$x \implies (x \vee y)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(c) **Tautology**

x	y	$x \vee y$	$x \vee \neg y$	$(x \vee y) \vee (x \vee \neg y)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(d) **Tautology**

x	y	$x \implies y$	$x \implies \neg y$	$(x \implies y) \vee (x \implies \neg y)$
T	T	T	F	T
T	F	F	T	T
F	T	T	T	T
F	F	T	T	T

(e) **Neither**

x	y	$x \vee y$	$\neg(x \wedge y)$	$(x \vee y) \wedge (\neg(x \wedge y))$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

(f) **Contradiction**

x	y	$x \implies y$	$\neg x \implies y$	$\neg y$	$(x \implies y) \wedge (\neg x \implies y) \wedge (\neg y)$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	F
F	F	T	F	T	F

2 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x \in \mathbb{Z})\ ((2 \mid x \vee 3 \mid x) \implies 6 \mid x)$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) All real numbers are complex numbers. This is true, since any real number x can equivalently be written as $x + 0i$.

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take $a = x$ and $b = 0$.

3 Prove or Disprove

(a) $\forall n \in \mathbb{N}$, if n is odd then $n^2 + 2n$ is odd.

(b) $\forall x, y \in \mathbb{R}$, $\min(x, y) = (x + y - |x - y|)/2$.

(c) $\forall a, b \in \mathbb{R}$ if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.

(d) $\forall r \in \mathbb{R}$, if r is irrational then $r + 1$ is irrational.

(e) $\forall n \in \mathbb{N}^+$, $10n^2 > n!$.

Solution:

(a) **Answer**: True.
Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. Substituting into the expression n^2+2n, we get $(2k+1)^2+2\times(2k+1)$. Simplifying the expression yields $4k^2+8k+3$. This can be rewritten as $2\times(2k^2+4k+1)+1$. Since $2k^2+4k+1$ is a natural number, by the definition of odd numbers, n^2+2n is odd.
Alternatively, we could also factor the expression to get $n(n+2)$. Since n is odd, $n+2$ is also odd. The product of 2 odd numbers is also an odd number. Hence n^2+2n is odd.

(b) **Answer**: True.
Proof: We will use a proof by cases. We know the following about the absolute value function for real number z.

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x-y| = y-x$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-y+x}{2} = x = \min(x,y).$$

Case 2: $x \geq y$. This means $|x-y| = x-y$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-x+y}{2} = y = \min(x,y).$$

(c) **Answer**: True.
Proof: We will use a proof by contraposition. Suppose that $a > 7$ and $b > 3$ (note that this is equivalent to $\neg\big(a \leq 7 \vee b \leq 3\big)$). Since $a > 7$ and $b > 3$, $a+b > 10$ (note that $a+b > 10$ is equivalent to $\neg\big(a+b \leq 10\big)$). Thus, if $a+b \leq 10$, then $a \leq 7$ or $b \leq 3$ (or both, as "or" is not "exclusive or" in this case).

(d) **Answer**: True.
Proof: We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form a/b where a and b are integers. Then r can be written as $(a-b)/b$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers. By contraposition, if r is irrational, then $r+1$ is irrational.

(e) **Answer**: False.
Proof: We will use proof by counterexample. Let $n = 6$. $10\times 6^2 = 360$. $6! = 720$. Since $10n^2 < n!$, the claim is false.

4 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $\forall x,\ x \in X \implies x \in Y$.

1. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
2. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
3. $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.
4. $f(A \cup B) = f(A) \cup f(B)$.
5. $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.
6. $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

1. Suppose x is such that $f(x) \in A \cup B$. Then either $f(x) \in A$, in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$. This proves that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

   Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

2. Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

   Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

3. Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

   Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

4. Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

   Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

5. Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

   Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

6. Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

   Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.