CS 70 Discrete Mathematics and Probability Theory
Summer 2017 Lu, Moulos, and Tang

HW 1

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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2 Tautologies and Contradictions

A *tautology* is an expression that evaluates to True for all possible combinations of its variables. A *contradiction* is an expression that evaluates to False for all possible combinations of its variables. State whether the following expressions are tautologies, contradictions, or neither. Justify your answers.

(a) $(x \vee y) \vee (x \vee \neg y)$

(b) $(x \vee y) \wedge (\neg(x \wedge y))$

(c) $x \wedge (x \implies y) \wedge (\neg y)$

(d) $x \implies (x \vee y)$

(e) $(x \implies y) \vee (x \implies \neg y)$

(f) $(x \implies y) \wedge (\neg x \implies y) \wedge (\neg y)$

Solution:

(a) **Tautology**

x	y	$x \lor y$	$x \lor \neg y$	$(x \lor y) \lor (x \lor \neg y)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(b) **Neither**

x	y	$x \lor y$	$\neg(x \land y)$	$(x \lor y) \land (\neg(x \land y))$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

(c) **Contradiction**

x	y	$x \implies y$	$\neg x \implies y$	$\neg y$	$(x \implies y) \land (\neg x \implies y) \land (\neg y)$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	F
F	F	T	F	T	F

(d) **Tautology**

x	y	$x \lor y$	$x \implies (x \lor y)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(e) **Tautology**

x	y	$x \implies y$	$x \implies \neg y$	$(x \implies y) \lor (x \implies \neg y)$
T	T	T	F	T
T	F	F	T	T
F	T	T	T	T
F	F	T	T	T

(f) **Contradiction**

x	y	$x \implies y$	$\neg x \implies y$	$\neg y$	$(x \implies y) \land (\neg x \implies y) \land (\neg y)$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	F
F	F	T	F	T	F

3 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x \in \mathbb{Z})\ ((2 \mid x \vee 3 \mid x) \implies 6 \mid x)$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) All real numbers are complex numbers. This is true, since any real number x can equivalently be written as $x + 0i$.

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take $a = x$ and $b = 0$.

4 Contraposition

Consider the statement "if $a + b < c + d$, then $a < c$ or $b < d$".

(a) Prove this statement with a direct proof.

(b) Prove this statement via contraposition.

(c) Which proof type was easier?

Solution:

(a) We can prove this with a bit of casework. First, in the case where $a < c$, we're immediately done, since we just need one of $a < c$ or $b < d$ to be true. The only other case is if $a \geq c$. In this case, we can say that $b = (a + b) - a < (c + d) - a \leq (c + d) - c = d$, and so we have that $b < d$. Thus, no matter which case we're in, we have that either $a < c$ or $b < d$.

(b) The implication we're trying to prove is $(a+b<c+d) \implies ((a<c) \vee (b<d))$, so the contrapositive is $((a \geq c) \wedge (b \geq d)) \implies (a+b \geq c+d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a+b \geq c+d$, which is exactly what we wanted.

(c) Either answer is acceptable here. The course staff thinks that the proof by contraposition is somewhat simpler and much easier to come up with. The point of this question is to reinforce that rewriting an implication as its contrapositive can often make it easier to come up with an elegant proof.

5 Pigeonhole Principle

(a) Prove the following statement: If you put $n+1$ balls into n bins, however you want, then at least one bin must contain at least two balls. This is known as the *pigeonhole principle*.

(b) Use the pigeonhole principle to show the following fact: If there are n students at a homework party ($n \geq 2$), then there are at least two students who are friends with exactly the same number of other students in the homework party. Assume that friendships are always mutual.

Solution:

(a) Suppose this is not the case. Then all the bins would contain at most one ball. Then the maximum number of balls we could have would be n, but this is a contradiction since we have $n+1$ balls.

(b) We will apply the pigeonhole principle in the following way: the n students are the "balls". The set $A = \{0, \ldots, n-1\}$ represents the possibilities for the number of friends that a given student has (each student is friends with between 0 and $n-1$ other students). The set A represents the "bins". At first, it seems like we're in trouble: A also has n elements, so the pigeonhole principle does not apply. However, notice that it is impossible for one student to have no friends *and* another student to be friends with everyone else. Therefore, the set of possible numbers of friends is actually either $\{0, \ldots, n-2\}$ or $\{1, \ldots, n-1\}$; in either case, we now have $n-1$ "bins" and by the pigeonhole principle, we conclude that at least two students have the same number of friends.

6 Airport

Suppose that there are $2n+1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, there is exactly one airplane departing from it, and heading towards the closest airport. Prove by induction that there is an airport which none of the airplanes are heading towards.

Solution:

For $n = 1$, let the 3 airports be A, B, C and let their distance be $|AB|, |AC|, |BC|$. Without loss of generality suppose $|BC| < |AB|$ and $|BC| < |AC|$. Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are heading towards airport A.

Now suppose the statement is proven for $n = k$, i.e. when there are $2k+1$ airports. For $n = k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports must be heading towards each other. Removing these two airports reduce the problem to $2k+1$ airports. From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights. When we add back the two airports that we removed, the airplane flights may change; in particular, it is possible that an airplane will now choose to fly to one of these two airports (because the airports that were added may be closer than the airport to which the airplane was previously flying), but observe that none of the airplanes will choose to fly towards the airport which had no incoming flights. Also, the two airports that were added back will have airplanes flying towards each other, so they too will not fly towards the airport with no incoming flights. We conclude that the airport which previously had no incoming flights will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k+1$. By induction, the claim holds for all $n \geq 3$.

7 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

Solution:

Prove by strong induction on n. Note that this is the first time students will have seen strong induction, so it is important that this problem be done in an interactive way that shows them how simple induction gets stuck.

- *Base Case:* $n = 1$ can be written with 1×2^0.
- *Inductive Hypothesis:* Assume that the statement is true for all $1 \leq k \leq n$.
- *Inductive Step:* If $n+1$ is divisible by 2, then it can use the representation of $(n+1)/2$.

$$(n+1)/2 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0$$
$$n+1 = 2 \cdot (n+1)/2 = c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \ldots + c_1 \cdot 2^2 + c_0 \cdot 2^1.$$

  Otherwise, n must be divisible by 2 and have $c_0 = 0$. We can obtain the representation of $n+1$ from n.

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + 0 \cdot 2^0$$
$$n+1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + 1 \cdot 2^0$$

Therefore, the statement is true.

8 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 1.\end{aligned}$$

Therefore, the statement is true. □

(b) **Claim:** For all negative integers n, $-1 - 3 - \cdots + (2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.
Inductive Hypothesis: Assume that $-1 - 3 - \cdots + (2n+1) = -n^2$.
Inductive Step: We need to prove that the statement is also true for $n-1$ if it is true for n, that is, $-1 - 3 - \cdots + (2(n-1)+1) = -(n-1)^2$. Starting from the left hand side,

$$\begin{aligned}-1 - 3 - \cdots + (2(n-1)+1) &= (-1 - 3 - \cdots + (2n+1)) + (2(n-1)+1) \\ &= -n^2 + (2(n-1)+1) \quad \text{(Inductive Hypothesis)} \\ &= -n^2 + 2n - 1 \\ &= -(n-1)^2.\end{aligned}$$

Therefore, the statement is true. □

(c) **Claim:** For all positive integers n, $\sum_{i=0}^{n} 2^{-i} \leq 2$.

Proof. We will prove a stronger statement, that is, $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$, by induction on n.
Base Case: $n = 1 \leq 2 - 1$. It is true for $n = 1$.

Inductive Hypothesis: Assume that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.
Inductive Step: We must show that $\sum_{i=0}^{n+1} 2^{-i} = 2 - 2^{-(n+1)}$. Starting from the left hand side,

$$\begin{aligned}\sum_{i=0}^{n+1} 2^{-i} &= \sum_{i=0}^{n} 2^{-i} + 2^{-(n+1)} \\ &= (2 - 2^{-n}) + 2^{-(n+1)} \quad \text{(Inductive Hypothesis)} \\ &= 2 - 2^{-(n+1)}.\end{aligned}$$

Since $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n} \leq 2$, the claim is true. □

(d) **Claim:** For all nonnegative integers n, $2n = 0$.

Proof. We will prove by strong induction on n.
Base Case: $2 \times 0 = 0$. It is true for $n = 0$.
Inductive Hypothesis: Assume that $2k = 0$ for all $0 \leq k \leq n$.
Inductive Step: We must show that $2(n+1) = 0$. Write $n+1 = a+b$ where $0 < a, b \leq n$. From the inductive hypothesis, we know $2a = 0$ and $2b = 0$, therefore,

$$2(n+1) = 2(a+b) = 2a + 2b = 0 + 0 = 0.$$

The statement is true. □

(e) **Claim:** Every positive integer $n \geq 2$ has a unique prime factorization.
In other words, let $2 \leq p_1, p_2, \ldots, p_i \leq n$ be all prime numbers that divide n, there is only one unique way to write n as a product of primes,

$$n = p_1^{d_1} \cdot p_2^{d_2} \cdots p_i^{d_i},$$

where $d_1, d_2, \ldots, d_i \in \mathbb{N}$.

Proof. We will prove by strong induction on n.
Base Case: 2 is a prime itself. It is true for $n = 2$.
Inductive Hypothesis: Assume that the statement is true for all $2 \leq k \leq n$.
Inductive Step: We must prove that the statement is true for $n+1$. If $n+1$ is prime, then it itself is a unique prime factorization. Otherwise, $n+1$ can be written as $x \times y$ where $2 \leq x, y \leq n$. From the inductive hypothesis, both x and y have unique prime factorizations. The product of unique prime factorizations is unique, therefore, $n+1$ has a unique prime factorization. □

Solution:

(a) Note that n is a real number. The proof is incorrect because it does not consider $0 < n < 1$, for which the claim is false. Also, by the way it is set up, it can only cover integers for $n \geq 1$.

(b) The proof is correct. The base case starts from the correct, identifiable end point, then the inductive step successfully proves that the statement continues to be true towards $-\infty$.

(c) The proof only has one minor error. It uses the wrong value for the base case $n = 1$. It should have shown that $2^0 + 2^{-1} = 2 - 2^{-1}$, instead of just using the value of n.

(d) The proof is incorrect. When $n = 0$, we cannot write $n+1 = 1 = a+b$ where $0 < a, b \leq n = 0$.

(e) The proof is incomplete because it fails to mention that more than one pair of x and y can satisfy $n+1 = x \times y$. For example, let $n+1 = 32$. (x, y) can be either $(2^2, 2^3)$ or $(2^1, 2^4)$. The proof should have addressed that there can be multiple pairs of (x, y), and shown that they will all lead to the same unique prime factorization (2^5 in case of 32).

    Neglecting possible cases can lead to wrong result. It just so happens that this claim is true and the uncovered case does not change the result.