CS 70 Discrete Mathematics and Probability Theory
Fall 2018 Alistair Sinclair and Yun Song HW 2

1 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n+1.\end{aligned}$$

Therefore, the statement is true. □

(b) **Claim:** For all negative integers n, $(-1) + (-3) + \cdots + (2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.
Inductive Hypothesis: Assume that $(-1) + (-3) + \cdots + (2n+1) = -n^2$.
Inductive Step: We need to prove that the statement is also true for $n-1$ if it is true for n, that is, $(-1) + (-3) + \cdots + (2(n-1)+1) = -(n-1)^2$. Starting from the left hand side,

$$\begin{aligned}(-1) + (-3) + \cdots + (2(n-1)+1) &= ((-1) + (-3) + \cdots + (2n+1)) + (2(n-1)+1) \\ &= -n^2 + (2(n-1)+1) \quad \text{(Inductive Hypothesis)} \\ &= -n^2 + 2n - 1 \\ &= -(n^2 - 2n + 1) \\ &= -(n-1)^2.\end{aligned}$$

Therefore, the statement is true. □

(c) **Claim:** For all nonnegative integers n, $2n = 0$.

Proof. We will prove by strong induction on n.
Base Case: $2 \times 0 = 0$. It is true for $n = 0$.
Inductive Hypothesis: Assume that $2k = 0$ for all $0 \leq k \leq n$.
Inductive Step: We must show that $2(n+1) = 0$. Write $n+1 = a+b$ where $0 < a,b \leq n$. From the inductive hypothesis, we know $2a = 0$ and $2b = 0$, therefore,

$$2(n+1) = 2(a+b) = 2a+2b = 0+0 = 0.$$

The statement is true. $\square$

Solution:

(a) Note that n is a real number. The proof is incorrect because it does not consider $0 < n < 1$, for which the claim is false. Also, by the way it is set up, it can only cover integers for $n \geq 1$.

(b) The proof is correct. The base case starts from the correct, identifiable end point, then the inductive step successfully proves that the statement continues to be true towards $-\infty$.

(c) The proof is incorrect. When $n = 0$, we cannot write $n+1 = 1 = a+b$ where $0 < a,b \leq n = 0$.

2 A Coin Game

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

Solution:

We can prove this by strong induction on n.

Base Case: If $n = 1$, you start with a stack of one coin, so the game immediately terminates. Your total score is zero–and indeed, $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$.

Inductive Hypothesis: Suppose that if you start with i coins (for i between 1 and n inclusive), your score will be $\frac{i(i-1)}{2}$ no matter what strategy you employ.

Inductive Step: Now suppose you start with $n+1$ coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks s_1 and s_2 (so $s_1 + s_2 = n+1$ and $s_1, s_2 \geq 1$). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get $s_1 s_2$ points from the first split. From the inductive hypothesis (which we can apply because s_1 and s_2 are between 1 and n), we know that the total number of points you get from future splits of stack 1 is $\frac{s_1(s_1-1)}{2}$ and similarly that the total number of points you get from future splits of stack 2 is $\frac{s_2(s_2-1)}{2}$, regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned} s_1 s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1) + 2s_1 s_2 + s_2(s_2-1)}{2} \\ &= \frac{\big(s_1(s_1-1) + s_1 s_2\big) + \big(s_2(s_2-1) + s_1 s_2\big)}{2} \\ &= \frac{s_1(s_1+s_2-1) + s_2(s_1+s_2-1)}{2} \\ &= \frac{(s_1+s_2)(s_1+s_2-1)}{2} \end{aligned}$$

Since $s_1 + s_2 = n+1$, this works out to $\frac{(n+1)(n+1-1)}{2}$, which is what we wanted to show your total number of points came out to. This completes our proof by induction.

3 Grid Induction

Pacman is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes). Every second he does one of the following (if possible):

(i) Walk one step down, to $(i, j-1)$.

(ii) Walk one step left, to $(i-1, j)$.

For example, if he is at $(5,0)$, his only option is to walk left to $(4,0)$; if Pacman is instead at $(3,2)$, he could walk either to $(2,2)$ or $(3,1)$.

Prove by induction that no matter how he walks, he will always reach $(0,0)$ in finite time. (*Hint*: Try starting Pacman at a few small points like $(2,1)$ and looking all the different paths he could take to reach $(0,0)$. Do you notice a pattern?)

Solution:

Following the hint, we notice that it seems as though Pacman takes $i+j$ seconds to reach $(0,0)$ if he starts in position (i, j), regardless of what path he takes. This would imply that he reaches $(0,0)$ in a finite amount of time since $i+j$ is a finite number. Thus, if we can prove this stronger statement, we'll also have proved that Pacman reaches $(0,0)$ in finite time. In order to simplify the induction, we will induct on the quantity $i+j$ rather than inducting on i and j separately.

Base Case: If $i+j=0$, we know that $i=j=0$, since i and j must be non-negative. Hence, we have that Pacman is already at position $(0,0)$ and so will take $0 = i+j$ steps to get there.

Inductive Hypothesis: Suppose that if Pacman starts at position (i,j) such that $i+j=n$, he will reach $(0,0)$ in n seconds regardless of his path.

Inductive Step: Now suppose Pacman starts at position (i,j) such that $i+j=n+1$. If Pacman's first move is to position $(i-1,j)$, the sum of his x and y positions will be $i-1+j=(i+j)-1=n$. Thus, our inductive hypothesis tells us that it will take him n further seconds to get to $(0,0)$ no matter what path he takes. If Pacman's first move isn't to $(i-1,j)$, then it must be to $(i,j-1)$. Again in this case, the inductive hypothesis will tell us that Pacman will use n more moves to get to $(0,0)$ no matter what path he takes. Thus, in either case, we have that Pacman will take a total of $n+1$ seconds (one for the first move and n for the remainder) in order to reach $(0,0)$, proving the claim for $n+1$.

One can also prove this statement without strengthening the inductive hypothesis. The proof isn't quite as elegant, but is included here anyways for reference. We first prove by induction on i that if Pacman starts from position $(i,0)$, he will reach $(0,0)$ in finite time.

Base Case: If $i=0$, Pacman starts at position $(0,0)$, so he doesn't need any more steps. Thus, it takes Pacman 0 steps to reach the origin, where 0 is a finite number.

Inductive Hypothesis: Suppose that if $i=n$ (that is, if Pacman starts at position $(n,0)$), he will reach $(0,0)$ in finite time.

Inductive Step: Now say Pacman starts at position $(n+1,0)$. Since he is on the x-axis, he has only one move: he has to move to $(n,0)$. From the inductive hypothesis, we know he will only take finite time to get to $(0,0)$ once he's gotten to $(n,0)$, so he'll only take a finite amount of time plus one second to get there from $(n+1,0)$. A finite amount of time plus one second is still a finite amount of time, so we've proved the claim for $i=n+1$.

We can now use this statement as the base case to prove our original claim by induction on j.

Base Case: If $j=0$, Pacman starts at position $(i,0)$ for some $i\in\mathbb{N}$. We proved above that Pacman must reach $(0,0)$ in finite time starting from here.

Inductive Hypothesis: Suppose that if Pacman starts in position (i,n), he'll reach $(0,0)$ in finite time no matter what i is.

Inductive Step: We now consider what happens if Pacman starts from position $(i,n+1)$, where i can be any natural number. If Pacman starts by moving down, we can immediately apply the inductive hypothesis, since Pacman will be in position (i,n). However, if Pacman moves to the left, he'll be in position $(i-1,n+1)$, so we can't yet apply the inductive hypothesis. But note that Pacman can't keep moving left forever: after i such moves, he'll hit the wall on the y-axis and be forced to move down. Thus, Pacman must make a vertical move after only finitely many horizontal moves–and once he makes that vertical move, he'll be in position (k,n) for some $0\le k\le i$, so the inductive hypothesis tells us that it will only take him a finite amount of time to reach $(0,0)$ from there. This means that Pacman can only take a finite amount of time moving to the left, one second making his first move down, then a finite amount of additional time after his first vertical move. Since a finite number plus one plus another finite number is still finite, this gives us our desired claim: Pacman must reach $(0,0)$ in finite time if he starts from position $(i, n+1)$ for any $i \in \mathbb{N}$.

4 Stable Marriage

Consider a set of four men and four women with the following preferences:

men	preferences	women	preferences
A	1>2>3>4	1	D>A>B>C
B	1>3>2>4	2	A>B>C>D
C	1>3>2>4	3	A>B>C>D
D	3>1>2>4	4	A>B>D>C

(a) Run on this instance the stable matching algorithm presented in class. Show each day of the algorithm, and give the resulting matching, expressed as $\{(M,W),\ldots\}$.

(b) Suppose we relax the rules for the men, so that each unpaired man proposes to the next woman on his list at a time of his choice (some men might procrastinate for several days, while others might propose and get rejected several times in a single day). Prove that this modification will not change what pairing the algorithm outputs.

Solution:

(a) The situations on the successive days are:
Day 1 Proposals: $A \to 1$, $B \to 1$, $C \to 1$, $D \to 3$; B and C are rejected.
Day 2 Proposals: $A \to 1$, $B \to 3$, $C \to 3$, $D \to 3$; C and D are rejected.
Day 3: Proposals: $A \to 1$, $B \to 3$, $C \to 2$, $D \to 1$; A is rejected.
Day 4: Proposals: $A \to 2$, $B \to 3$, $C \to 2$, $D \to 1$; C is rejected.
Day 5: Proposals: $A \to 2$, $B \to 3$, $C \to 4$, $D \to 1$; no one is rejected.
Final matching: $(A,2),(B,3),(C,4),(D,1)$.

(b) Assume that when a proposal is made and an answer is received, we write it down on a list L and enumerate them. Now the proof is similar to the proof in class that the algorithms finds the male optimal pairing.

We should prove that the pairing P' that results is the same as pairing P found by the regular algorithm. Assume the opposite, so either there is a woman that rejects a man who was her pair in P or there is a woman who does not receive a proposal from a man who was her pair in P. By the Well Ordering Principle, there is the first entry in the list L for one of those happens; let W denote the woman for whom it happens and let M denote her pair in P.

First, if W rejected M then it is because she got a proposal from another man M' who is higher in her preference list. But M' proposed to her only because he was rejected by another woman W' who is higher in his preference list and did not reject him in the original algorithm. In other words, even before W rejected M, W' rejected her previous pair M'. This is a contradiction because our argument relies on the fact that W is the first woman to reject her partner M from the pairing P. However, we just showed that W' rejected M' (her partner from P) even before that.

Second, if W did not get a proposal from M then it is because M was not rejected by some other woman W' who is higher in his preference list but who rejected him in the original algorithm. This can only happen if W' never receives a proposal from M', her partner in P, or if W' rejects M' in favor of M. The latter would imply that (M, W') is a rogue couple in P, which is a contradiction that P is a stable pairing. The former would contradict our assumption that W is the first woman to not receive a proposal from her partner in P.

Therefore, the resulting pairing P' is the same as P.

5 Optimal Partners

In the notes, we proved that the Stable Marriage Algorithm always outputs the male-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every man with his best choice results in a pairing at all. Prove by contradiction that no two men can have the same optimal partner. (Note: your proof should not rely on the fact that the Stable Marriage Algorithm outputs the male-optimal pairing.)

Solution:

For the sake of contradiction, assume that we have some instance of the Stable Marriage problem where both man M and man M' have woman W as their optimal partner. We further assume without loss of generality that W prefers M to M' (if this is not the case, we can just switch the names to make it so). Because W is M''s optimal partner, we know by definition that there must exist some stable pairing P in which M' is paired with W. Call M's partner in P W^*. Since W is M's optimal partner, we know by definition that M must prefer W to any woman he is ever paired with in any stable pairing–including W^*. As well, we previously said that W prefers M to M'. Thus, M and W would form a rogue couple in P, which is a contradiction because P is stable. So our initial assumption must be false: there must never exist two men who have the same optimal partner.

6 Examples or It's Impossible

Determine if each of the situations below is possible with the traditional propose-and-reject algorithm. If so, give an example with at least 3 men and 3 women. Otherwise, give a brief proof as to why it's impossible.

(a) Every man gets his first choice.

(b) Every woman gets her first choice, even though her first choice does not prefer her the most.

(c) Every woman gets her last choice.

(d) Every man gets his last choice.

(e) A man who is second on every woman's list gets his last choice.

Solution:

(a) One way to construct an example is to have each man have a distinct first choice. For example,

Men	Preferences
1	$A > C > B$
2	$B > C > A$
3	$C > A > B$

Women	Preferences
A	$3 > 2 > 1$
B	$1 > 3 > 2$
C	$2 > 1 > 3$

(b) An example where every woman gets her first choice, and every man his second choice:

Men	Preferences
1	$C > A > B$
2	$A > B > C$
3	$A > C > B$

Women	Preferences
A	$1 > 2 > 3$
B	$2 > 3 > 1$
C	$3 > 1 > 2$

(c) One method for constructing an example of this is to have each woman have a unique last choice, and have each man most prefer the woman who likes him the least. As an example,

Men	Preferences
1	$A > C > B$
2	$B > C > A$
3	$C > B > A$

Women	Preferences
A	$2 > 3 > 1$
B	$1 > 3 > 2$
C	$1 > 2 > 3$

(d) Impossible. By contradiction: On the last day, every man proposes to his unique least-favorite woman. So prior to the last day, every man has been rejected by all $n-1$ other women, meaning every woman must have rejected all $n-1$ other men. But we can prove that there must exist at least one woman who only ever receives one proposal (we defer this proof to the next paragraph). A woman who receives only one proposal cannot reject any men, so it is impossible for every woman to have rejected $n-1$ men.

To complete this proof, we now just have to show that there always exists a woman who gets only one proposal throughout the entire course of the algorithm. For each woman w, let d_w be the first day in the algorithm that w received a proposal, and let D be the maximum d_w for any woman w. By the improvement lemma, we know that if a woman receives a proposal on day i, she must also receive a proposal on day j for all $j \geq i$. Since $D \geq d_w$ for all women w, this means that all women must get a proposal on day D, so the algorithm must terminate on day D. But there must be some woman w^* for which $d_{w^*} = D$ (as D is the maximum of the d_ws), meaning that the first day w^* got a proposal was also the last day of the algorithm. Hence, w^* only received one proposal: the one she got on day D.

(e) One method for constructing an example is to have m_i and w_i both prefer each other the most for $1 \leq i < n$. We can then put m_n in the second position on every woman's list and w_n at the last position on m_n's list. An explicit example of this with $n = 3$ is shown below.

Men	Preferences
1	$A > C > B$
2	$B > C > A$
3	$A > B > C$

Women	Preferences
A	$1 > 3 > 2$
B	$2 > 3 > 1$
C	$1 > 3 > 2$