CS 70 Summer 2018 — Discrete Mathematics and Probability Theory — Sinho Chewi and Vrettos Moulos — HW 2

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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1 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 1.\end{aligned}$$

Therefore, the statement is true. □

(b) **Claim:** For all negative integers n, $-1 - 3 - \cdots + (2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.

Inductive Hypothesis: Assume that $-1-3-\cdots+(2n+1)=-n^2$.
Inductive Step: We need to prove that the statement is also true for $n-1$ if it is true for n, that is, $-1-3-\cdots+(2(n-1)+1)=-(n-1)^2$. Starting from the left hand side,

$$\begin{aligned}
-1-3-\cdots+(2(n-1)+1) &= (-1-3-\cdots+(2n+1))+(2(n-1)+1) \\
&= -n^2+(2(n-1)+1) \quad \text{(Inductive Hypothesis)} \\
&= -n^2+2n-1 \\
&= -(n-1)^2.
\end{aligned}$$

Therefore, the statement is true. □

(c) **Claim:** For all nonnegative integers n, $2n=0$.

Proof. We will prove by strong induction on n.
Base Case: $2\times 0=0$. It is true for $n=0$.
Inductive Hypothesis: Assume that $2k=0$ for all $0\le k\le n$.
Inductive Step: We must show that $2(n+1)=0$. Write $n+1=a+b$ where $0<a,b\le n$. From the inductive hypothesis, we know $2a=0$ and $2b=0$, therefore,

$$2(n+1)=2(a+b)=2a+2b=0+0=0.$$

The statement is true. □

Solution:

(a) Note that n is a real number. The proof is incorrect because it does not consider $0<n<1$, for which the claim is false. Also, by the way it is set up, it can only cover integers for $n\ge 1$.

(b) The proof is correct. The base case starts from the correct, identifiable end point, then the inductive step successfully proves that the statement continues to be true towards $-\infty$.

(c) The proof is incorrect. When $n=0$, we cannot write $n+1=1=a+b$ where $0<a,b\le n=0$.

2 A Coin Game

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select a stack of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3\cdot 2=6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

Solution:

We can prove this by strong induction on n.

Base Case: If $n = 1$, you start with a stack of one coin, so the game immediately terminates. Your total score is zero–and indeed, $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$.

Inductive Hypothesis: Suppose that if you start with i coins (for i between 1 and n), your score will be $\frac{i(i-1)}{2}$ no matter what strategy you employ.

Inductive Step: Now suppose you start with $n+1$ coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks s_1 and s_2 (so $s_1 + s_2 = n+1$ and $s_1, s_2 \geq 1$). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get $s_1 s_2$ points from the first split. From the inductive hypothesis (which we can apply because s_1 and s_2 are between 1 and n), we know that the total number of points you get from future splits of stack 1 is $\frac{s_1(s_1-1)}{2}$ and similarly that the total number of points you get from future splits of stack 2 is $\frac{s_2(s_2-1)}{2}$. Thus, the total number of points we score is

$$\begin{aligned}
s_1 s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1) + 2s_1 s_2 + s_2(s_2-1)}{2} \\
&= \frac{\big(s_1(s_1-1) + s_1 s_2\big) + \big(s_2(s_2-1) + s_1 s_2\big)}{2} \\
&= \frac{s_1(s_1+s_2-1) + s_2(s_1+s_2-1)}{2} \\
&= \frac{(s_1+s_2)(s_1+s_2-1)}{2}
\end{aligned}$$

Since $s_1 + s_2 = n+1$, this works out to $\frac{(n+1)(n+1-1)}{2}$, which is what we wanted to show your total number of points came out to. This completes our proof by induction.

3 Calculator Enigma

Suppose you have a calculator on which the only working keys are $3, 6, 9, (,), +, *, -$, and the decimal point. Prove that the only numbers you can make this calculator display are of the form $\frac{3k}{10^\ell}$ for $k \in \mathbb{Z}, \ell \in \mathbb{N}$; that is, show that the only numbers you can display are multiples of three with a decimal point inserted somewhere.

Hint: Use the well-ordering principle. What set might it be useful to define a well-ordering on?

Solution:

We can prove this using the well-ordering principle. Let E be the set of all valid expressions you can enter into the calculator. (This means E will exclude things that don't parse like ")$3*.($"

and also anything that uses any of the keys that are broken). We can order the expressions in E by length, breaking ties alphabetically. If there exist expressions in E which don't evaluate to something of the form $\frac{3k}{10^\ell}$, the well ordering principle tells us that there must be some smallest expression e like that. We consider two possible cases:

1. Suppose e does not contain the symbols $+$, $*$, or $-$. This means that e consists of only a single number made up of the digits 3, 6, and 9, possibly with a decimal point. Such numbers can always be written in the form $\frac{3k}{10^\ell}$ for the appropriate choice of k and ℓ, so our e cannot be of this form.

2. If e contains $+$, $*$, or $-$, there must be one of those symbols that gets evaluated last. What this means is that we, in effect, evaluate everything to the left of that symbol, separately evaluate everything to the right of the symbol, then add / multiply / subtract the two results. Let e_1 be the expression to the left and let e_2 be the expression to the right. Since e_1 and e_2 are smaller than e and e is the smallest expression that doesn't evaluate to something of the form $\frac{3k}{10^\ell}$, we know that there exist some numbers $k_1, k_2 \in \mathbb{Z}$ and $\ell_1, \ell_2 \in \mathbb{N}$ such that e_1 evaluates to $\frac{3k_1}{10^{\ell_1}}$ and e_2 evaluates to $\frac{3k_2}{10^{\ell_2}}$. If we add, multiply, or subtract two numbers of this form, we'll get back another number of that form (for example, if we multiplied e_1 and e_2, we would get $\frac{3(3k_1k_2)}{10^{\ell_1+\ell_2}}$). This contradicts that e was not of the form $\frac{3k}{10^\ell}$, so we have that this case also cannot provide a counterexample.

Since we reached a contradiction no matter which case applies, we can conclude that there is no smallest expression e that does not evaluate to something of the correct form. Hence, we can conclude that all valid expressions we might enter will evaluate to something of the form $\frac{3k}{10^\ell}$.

4 Build-Up Error?

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger build-up error.

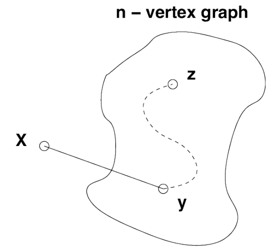
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$.

Solution:

The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex"*. Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —— and properly so, since the claim is false!

5 Proofs in Graphs

Please prove or disprove the following claims.

(a) In Old California, all roads were one way streets. Suppose Old California had n cities ($n \geq 2$) such that for every pair of cities X and Y, either X had a road to Y or Y had a road to X. Prove or disprove that there existed a city which was reachable from every other city by traveling through at most 2 roads.

[*Hint:* Induction]

(b) In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree.

Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove or disprove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

Solution:

(a) We prove this by induction on the number of cities n.

Base case For $n = 2$, there's always a road from one city to the other.

Inductive Hypothesis When there are k cities, there exists a city c that is reachable from every other city by traveling through at most 2 roads.

Inductive Step Consider the case where there are $k+1$ cities. Remove one of the cities d and all of the roads to and from d. Now there are k cities, and by our inductive hypothesis, there exists some city c which is reachable from every other city by traveling through at most 2 roads. Let A be the set of cities with a road to c, and B be the set of cities two roads away from c. The inductive hypothesis states that the set S of the k cities consists of $S = \{c\} \cup A \cup B$.

Now add back d and all roads to and from d. Between d and every city in S, there must be a road from one to the other. If there is at least one road from d to $\{c\} \cup A$, c would still be reachable from d with at most 2 road traversals. Otherwise, if all roads from $\{c\} \cup A$ point to d, d will be reachable from every city in B with at most 2 road traversals, because every city in B can take one road to go to a city in A, then take one more road to go to d. In either case there exists a city in the new set of $k+1$ cities that is reachable from every other city by traveling at most 2 roads.

(b) We split the $2m$ odd-degree vertices into m pairs, and join each pair with an edge, adding m more edges in total. (Here, we allow for the possibility of multi-edges, that is, pairs of vertices with more than one edge between them.) Notice that now all vertices in this graph are of even degree. Now by Euler's theorem the resulting graph has an Eulerian tour. Removing the m added edges breaks the tour into m walks covering all the edges in the original graph, with each edge belonging to exactly one walk.

6 Always, Sometimes, or Never

In each part below, you are given some information about a graph G. Using only the information in the current part, say whether G will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) G can be vertex-colored with 4 colors.

(b) G requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of G and v is the number of vertices of G.

(d) G is connected, and each vertex in G has degree at most 2.

(e) Each vertex in G has degree at most 2.

Solution:

(a) Either planar or non-planar. By the 4-color theorem, any planar graph can provide the planar example. The easiest non-planar example is $K_{3,3}$, which can be 2-colored because it is bipartite. (Certainly, any graph which can be colored using only 2 colors can also be colored using 4 colors.)

(b) Always non-planar. The 4-color theorem tells us that if a graph is planar, it can be colored using only 4 colors. The contrapositive of this is that if a graph requires more than 4 colors to vertex-color, it must be non-planar. (Using the 5- or 6-color theorem would also work.)

(c) Either planar or non-planar. From the notes, we know that every planar graph follows this formula, so any planar graph is a valid planar example. The easiest non-planar example is again $K_{3,3}$, which has $e = 9$ and $v = 6$, meaning our formula becomes $9 \leq 3(6) - 6 = 12$, which is certainly true.

(d) Always planar. There are two cases to deal with here: either G is a tree, or G is not a tree and so contains at least one cycle. In the former case, we're immediately done, since all trees are planar. In the latter case, consider any cycle in G. We know that every vertex in that cycle is adjacent to the vertex to its left in the cycle and to the vertex to its right in the cycle. But we also know that no vertex can be connected to more than two other vertices, so the cycle isn't connected to anything else. But G is a connected graph, so we must have that G is just a single large cycle. And we can certainly draw a simple cycle on a plane without crossing any edges, so even in this case G is still planar.

(e) Always planar. Each of G's connected components is connected and has no vertex of degree more than 2, so by the previous part, each of them must be planar. Thus, each of G's connected components must be planar, so G itself must be planar.

7 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L to a vertex in R (so there does not exist an edge that connects two vertices in L or two vertices in R).

(a) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Prove that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$.

(b) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Let s and t denote the average degree of vertices in L and R respectively. Prove that $s/t = |R|/|L|$.

(c) A double of a graph G consists of two copies of G with edges joining the corresponding "mirror" points. Now suppose that G_1 is a bipartite graph, G_2 is a double of G_1, G_3 is a double of G_2, and so on. (Each G_{i+1} has twice as many vertices as G_i). Show that $\forall n \geq 1$, G_n is bipartite.

Solution:

(a) Since G is bipartite, each edge connects one vertex in L with a vertex in R. Since each edge contributes equally to $\sum_{v \in L} \deg(v)$ and $\sum_{v \in R} \deg(v)$, we see that these two values must be equal.

(b) By part (a), we know that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$. Thus $|L| \cdot s = |R| \cdot t$. A little algebra gives us the desired result.

(c) We use induction. Let $P(n)$ be the proposition that G_n is bipartite. The base case is when $n = 1$. We see that $P(1)$ must be true by definition (and construction) of G_1. Now suppose that for $k \geq 1$, $P(k)$ holds. We see that the graph G_{k+1} consists of two subgraphs, each having the same structure as G_k, except the edges joining the corresponding edges of the two subgraphs. Remove the extra edges. Since $P(k)$ is true, we can label the two subgraphs into disjoint sets $\{L_1, R_1\}$ and $\{L_2, R_2\}$. Then we can define new sets $L = \{L_1, R_2\}$ and $R = \{R_1, L_2\}$ that are disjoint. Every edge connects a vertex from L_1 to R_1 and from L_2 to R_2, so it connects from L to R. Adding back all the removed edges, we see that each added edge connects a vertex from L_1 to L_2 and from R_1 to R_2, so every added edge connects from L to R. Thus, the remaining graph G_{k+1} is still bipartite.

8 Modular Arithmetic Solutions

Find all solutions (modulo the corresponding modulus) to the following equations. Prove that there are no other solutions (in a modular setting) to each equation.

(a) $2x \equiv 5 \pmod{15}$

(b) $2x \equiv 5 \pmod{16}$

(c) $5x \equiv 10 \pmod{25}$

Solution:

(a) Since 2 has an inverse modulo 15, this equation will have exactly one solution. We can get this solution by multiplying both sides by $2^{-1} \pmod{15}$, which is 8. This gives us that $x \equiv 40 \equiv 10 \pmod{15}$.

(b) Our trick from the last part doesn't work any more, since 2 is not relatively prime to 16, and thus doesn't have a multiplicative inverse mod 16. Indeed, this equation has no solutions at all, since the left side will always be even and the right side will always be odd. More formally, we know that $2x \equiv 5 \pmod{16}$ is equivalent to $2x + 16k = 5$ for some $k \in \mathbb{Z}$. We can factor out a two from the left hand side, showing that it is divisible by 2, so there is no choice of x and k that will make the equality hold.

(c) Again, we have that the coefficient on x does not have a multiplicative inverse in our chosen modulus. However, unlike the previous part, we can still find solutions to this equation. We can see by inspection that $x = 2$ will be one possible solution. Furthermore, if x is a solution to this equation, $x + 5$ will be as well, since $5(x+5) = 5x + 25 \equiv 5x \pmod{25}$. This tells us that $x = 7$, $x = 12$, $x = 17$, and $x = 22$ are also solutions to this equation.

We now just have to show that these are the only five solutions to this equation modulo 25. Suppose we have some solution x to the equation. This means that $5x \equiv 10 \pmod{25}$, or equivalently, that $5x + 25k = 10$ for some $k \in \mathbb{Z}$. This is now an equation over the real numbers, so we can divide the whole thing by 5, giving us that $x + 5k \equiv 2$. But now that's just definitionally saying that $x \equiv 2 \pmod{5}$. Thus, we have that all solutions to this equation must be equivalent to 2 modulo 5–and the only numbers modulo 25 that satisfy this condition are 2, 7, 12, 17, and 22. This tells us that indeed, these five numbers are the only possible values for x.