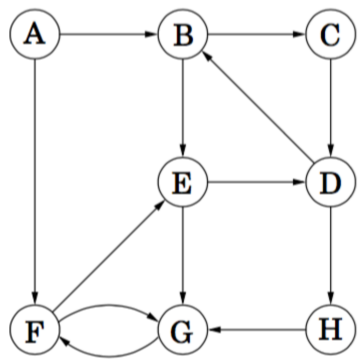
CS 70 Discrete Mathematics and Probability Theory
Fall 2020 **HW 3**

1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $(B,C),(C,D),(D,B)$
ii. $(F,G),(G,F)$
iii. $(A,B),(B,C),(C,D),(D,B)$
iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in G?

i. (E,G)
ii. $(E,G),(G,F)$
iii. $(F,G),(G,F)$
iv. $(A,B),(B,C),(C,D),(H,G)$
v. $(E,G),(G,F),(F,G),(G,C)$
vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in G?

i. (E,G)

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$V = \{A,B,C,D,E,F,G,H\}$,
$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),$
$(H,G)\}$.

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$(B,C),(C,D),(D,H),(H,G),(G,F)$
$(B,E),(E,D),(D,H),(H,G),(G,F)$
$(B,E),(E,G),(G,F)$
The first two have length 5, while the last one has length 3, so the last one is the shortest path.

(d) A cycle is a path that starts and ends at the same point. This means that (iii) is not a cycle, since it starts at A but ends at B. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)$ should be distinct, so (iv) is not a cycle. The correct answers are (i) and (ii).

(e) A walk consists of any sequence of edges such that the endpoint of each edge is the same as the starting vertex of the next edge in the sequence. Example (iv) does not fit this definition–even though it uses only valid edges, the endpoint of the second to last edge in D, while the start point of the next edge is H. Example (v) also is not a walk, since it tries to walk from G to C as its last step, but there is no such edge. All the rest are walks.

(f) A tour is simply a walk that has the same start and end vertex. Only (iii) satisfies this definition. Note in part (d), we already said that (iii) was a cycle–and indeed, all cycles are also tours.

(g) True. In order for there to be a cycle in G starting and ending at some vertex v, we would need at least two edges incident to v: one to leave v at the start of the cycle, and one to return to v at the end. If every vertex has degree at most 1, no vertex has two or more edges incident on it, so no vertex is capable of acting as the start and end point of a cycle.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If this process terminates, the part of our walk from the first time we visited w until the second time is a cycle. Thus, it remains only to argue this process always terminates.

Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree at least 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

2 Binary Trees

You have seen the recursive definition of binary trees from lecture and from previous classes. Here, we define binary trees in graph theoretic terms as follows (**Note:** here we will modify the definition of leaves slightly for consistency).

- A binary tree of height > 0 is a tree where exactly one vertex, called the **root**, has degree 2, and all other vertices have degrees 1 or 3. Each vertex of degree 1 is called a **leaf**. The **height** h is defined as the maximum length of the path between the root and any leaf.
- A binary tree of height 0 is the graph with a single vertex. The vertex is both a leaf and a root.

(a) Let T be a binary tree of height > 0, and let $h(T)$ denote it's height. Let r be the root in T and u and v be it's neighbors. Show that removing r from T will result in two binary trees, L, R with roots u and v respectively. Also, show that $h(T) = \max(h(L), h(R)) + 1$

(b) Using the graph theoretic definition of binary trees, prove that the number of vertices in a binary tree of height h is at most $2^{h+1} - 1$

(c) Prove that all binary trees with n leaves have $2n - 1$ vertices

Solution:

(a) Since r has degree 2, removing it will break T into two connected components, call them L and R. By symmetry, we just need to prove that L is a binary tree. Without loss of generality, suppose $u \in L$. Before removing r, u had degree 1 or 3. If u had degree 1, then after removing r, u is a single vertex, and so is a binary tree of height 0, and also is a root. If u had degree 3, then after removing r, u has degree 2, and all other vertices in L have degree 1 or 3. Thus, L is a binary tree with root u.

To prove that $h(T) = \max(h(L), h(R)) + 1$, we note that because T is a tree, any path from r to a leaf must go through either u or v but not both. Thus the maximum distance from r to any leaf is one plus either the maximum distance from u to any leaf in L (as the path cannot go back through r) or the maximum distance from v to any leaf in R. Formally, if we define $\mathscr{L}(L)$ and $\mathscr{L}(R)$ to be the set of leaves in L and R respectively, and $d(r,l)$ as the length of the path from r to some leaf l, then we have

$$\begin{aligned} h(T) &= \max(1 + \max_{l \in \mathscr{L}(L)}(d(u,l)), 1 + \max_{l \in \mathscr{L}(R)}(d(u,l))) \\ &= 1 + \max(h(L), h(R)) \end{aligned}$$

(b) Induction: **Base Case** a binary tree of height 0 is a singleton and so has $2^1 - 1 = 1$ vertex. **Inductive Hypothesis:** assume for all $k < h$, a binary tree of height k has at most $2^{k+1} - 1$ vertices. **Inductive Step:** By part a, we can remove the root from a binary tree and obtain two binary trees: L, and R of height k and l respectively. Since $h(T) = \max(h(L), h(R)) + 1$, we know that $k, l < h$ so we can apply the inductive hypothesis to L and R. Thus, we have that the number of vertices in T is at most $1 + 2^{k+1} - 1 + 2^{l+1} - 1 \leq 2^h * 2 - 1$.

(c) Induction: **Base Case** if a binary tree has one leaf, it is a singleton and so has $1 = 2 * 1 - 1$ vertices. **Inductive Hypothesis:** assume for all $k < n$, a binary tree with k leaves has $2k - 1$ vertices. **Inductive Step:** For a binary tree, T, with $n > 1$ leaves, remove the root, r and break T into binary trees L and R. Suppose L has a leaves and R has b leaves. Note that all the leaves of T are in L or R, as $n > 1$ implies the root is not a leaf, which means $a + b = n$. By the inductive hypothesis, L has $2a - 1$ vertices, and R has $2b - 1$ vertices, and so the number of vertices in T is $2a - 1 + 2b - 1 + 1 = 2(a + b) - 1 = 2n - 1$.

3 Proofs in Graphs

Please prove or disprove the following claims.

(a) On the axis from San Francisco traffic habits to Los Angeles traffic habits, Old California is more towards San Francisco: that is, civilized. In Old California, all roads were one way streets. Suppose Old California had n cities ($n \geq 2$) such that for every pair of cities X and Y, either X had a road to Y or Y had a road to X. Prove or disprove that there existed a city which was reachable from every other city by traveling through at most 2 roads.

[*Hint:* Induction]

(b) In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree.

Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove or disprove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

Solution:

(a) We prove this by induction on the number of cities n.

Base case For $n = 2$, there's always a road from one city to the other.

Inductive Hypothesis When there are k cities, there exists a city c that is reachable from every other city by traveling through at most 2 roads.

Inductive Step Consider the case where there are $k+1$ cities. Remove one of the cities d and all of the roads to and from d. Now there are k cities, and by our inductive hypothesis, there exists some city c which is reachable from every other city by traveling through at most 2 roads. Let A be the set of cities with a road to c, and B be the set of cities two roads away from c. The inductive hypothesis states that the set S of the k cities consists of $S = \{c\} \cup A \cup B$.

Now add back d and all roads to and from d. Between d and every city in S, there must be a road from one to the other. If there is at least one road from d to $\{c\} \cup A$, c would still be reachable from d with at most 2 road traversals. Otherwise, if all roads from $\{c\} \cup A$ point to d, d will be reachable from every city in B with at most 2 road traversals, because every city in B can take one road to go to a city in A, then take one more road to go to d. In either case there exists a city in the new set of $k+1$ cities that is reachable from every other city by traveling at most 2 roads.

(b) We split the $2m$ odd-degree vertices into m pairs, and join each pair with an edge, adding m more edges in total. (Here, we allow for the possibility of multi-edges, that is, pairs of vertices with more than one edge between them.) Notice that now all vertices in this graph are of even degree. Now by Euler's theorem the resulting graph has an Eulerian tour. Removing the m added edges breaks the tour into m walks covering all the edges in the original graph, with each edge belonging to exactly one walk.

4 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

(a) Assume toward contradiction that $K_{3,3}$ were planar. In $K_{3,3}$, there are $v = 6$ vertices and $e = 9$ edges. If $K_{3,3}$ were planar, from Euler's formula we would have $v - e + f = 2 \Rightarrow f = 5$. On the other hand, each region is bounded by at least four edges, so $4f \leq 2e$, i.e., $20 \leq 18$, which is a contradiction. Thus, $K_{3,3}$ is not planar.

(b) In this problem, we use proof by contradiction. Assume G is planar. Select any five vertices out of the seven. Consider the subgraph formed by these five vertices. They cannot form K_5, since G is planar. So some pair of vertices amongst these five has no edge between them. Label these vertices v_1 and v_2. The remaining five vertices of G besides v_1 and v_2 cannot form K_5 either, so there is a second pair of vertices amongst these new five that has no edge between them. Label these v_3 and v_4. Label the remaining three vertices v_5, v_6 nd v_7. Since $v_1 v_2$ is not an edge, by property T (which states any three vertices must have at least two edges between them)it must be that $\{v_1, v\}$ and $\{v_2, v\}$ are edges, where $v \in \{v_3, v_4, v_5, v_6, v_7\}$. Similarly for v_3, v_4 we have that $\{v_3, v\}$ and $\{v_4, v\}$ are edges, where $v \in \{v_1, v_2, v_5, v_6, v_7\}$. Now consider the subgraph induced by $\{v_1, v_2, v_3, v_5, v_6, v_7\}$. With the three vertices $\{v_1, v_2, v_3\}$ on one side and $\{v_5, v_6, v_7\}$ on the other, we observe that $K_{3,3}$ is a subgraph of this induced graph. This contradicts the fact that G is planar.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with greater than 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

5 Always, Sometimes, or Never

In each part below, you are given some information about the so-called original graph, OG. Using only the information in the current part, say whether OG will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) OG can be vertex-colored with 4 colors.

(b) OG requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of OG and v is the number of vertices of OG.

(d) OG is connected, and each vertex in OG has degree at most 2.

(e) Each vertex in OG has degree at most 2.

Solution:

(a) Either planar or non-planar. By the 4-color theorem, any planar graph can provide the planar example. The easiest non-planar example is $K_{3,3}$, which can be 2-colored because it is bipartite.

(Certainly, any graph which can be colored using only 2 colors can also be colored using 4 colors.)

(b) Always non-planar. The 4-color theorem tells us that if a graph is planar, it can be colored using only 4 colors. The contrapositive of this is that if a graph requires more than 4 colors to vertex-color, it must be non-planar. (Using the 5- or 6-color theorem would also work.)

(c) Either planar or non-planar. From the notes, we know that every planar graph follows this formula, so any planar graph is a valid planar example. The easiest non-planar example is again $K_{3,3}$, which has $e = 9$ and $v = 6$, meaning our formula becomes $9 \leq 3(6) - 6 = 12$, which is certainly true.

(d) Always planar. There are two cases to deal with here: either G is a tree, or G is not a tree and so contains at least one cycle. In the former case, we're immediately done, since all trees are planar. In the latter case, consider any cycle in G. We know that every vertex in that cycle is adjacent to the vertex to its left in the cycle and to the vertex to its right in the cycle. But we also know that no vertex can be connected to more than two other vertices, so the cycle isn't connected to anything else. But G is a connected graph, so we must have that G is just a single large cycle. And we can certainly draw a simple cycle on a plane without crossing any edges, so even in this case G is still planar.

(e) Always planar. Each of G's connected components is connected and has no vertex of degree more than 2, so by the previous part, each of them must be planar. Thus, each of G's connected components must be planar, so G itself must be planar.

6 Touring Hypercube

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- v_0 and v_k are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if n is even. (*Hint: Euler's theorem*)

(b) Show that every hypercube has a Hamiltonian tour.

Solution:

(a) In the n-dimensional hypercube, every vertex has degree n. If n is odd, then by Euler's Theorem there can be no Eulerian tour. On the other hand, the hypercube is connected: we can get from any one bit-string x to any other y by flipping the bits they differ in one at a time. Therefore, when n is even, since every vertex has even degree and the graph is connected, there is an Eulerian tour.

(b) By induction on n. When $n = 1$, there are two vertices connected by an edge; we can form a Hamiltonian tour by walking from one to the other and then back.

Let $n \geq 1$ and suppose the n-dimensional hypercube has a Hamiltonian tour. Let H be the $n+1$-dimensional hypercube, and let H_b be the n-dimensional subcube consisting of those strings with initial bit b.

By the inductive hypothesis, there is some Hamiltonian tour T on the n-dimensional hypercube. Now consider the following tour in H. Start at an arbitrary vertex x_0 in H_0, and follow the tour T except for the very last step to vertex y_0 (so that the next step would bring us back to x_0). Next take the edge from y_0 to y_1 to enter cube H_1. Next, follow the tour T in H_1 backwards from y_1, except the very last step, to arrive at x_1. Finally, take the step from x_1 to x_0 to complete the tour. By assumption, the tour T visits each vertex in each subcube exactly once, so our complete tour visits each vertex in the whole cube exactly once.

To build some intuition, here are the first few cases:

- $n = 1$: 0, 1
- $n = 2$: 00, 01, 11, 10 [Take the $n = 1$ tour in the 0-subcube (vertices with a 0 in front), move to the 1-subcube (vertices with 1 in front), then take the tour backwards. We know 10 connects to 00 to complete the tour.]
- $n = 3$: 000, 001, 011, 010, 110, 111, 101, 100 [Take the $n = 2$ tour in the 0-subcube, move to the 1-subcube, then take the tour backwards. We know 100 connects to 000 to complete the tour.]

The sequence produced with this method is known as a Gray code.