CS 70	Discrete Mathematics and Probability Theory	
Summer 2017	Hongling Lu, Vrettos Moulos, and Allen Tang	HW 4

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. (Signature here)

1 Solve the Rainbow

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

(d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?

(e) If you take three Skittles from the bag, what is the probability that they are all the same color?

Solution:

(a) We will use the law of total probability. Let G be the event that you take a green Skittles from the bag, A be the event that your roommate ate all of the green Skittles, H be the event that your roommate ate half the green Skittles, and F be the event that your roommate ate five green Skittles. Then, we get the total probability as following:

$$\mathbb{P}(G) = \mathbb{P}(G \cap A) + \mathbb{P}(G \cap H) + \mathbb{P}(G \cap F) \tag{1}$$
$$= \mathbb{P}(G \mid A)\mathbb{P}(A) + \mathbb{P}(G \mid H)\mathbb{P}(H) + \mathbb{P}(G \mid F)\mathbb{P}(F) \tag{2}$$
$$= 0 \cdot \frac{1}{2} + \frac{10}{90} \cdot \frac{1}{4} + \frac{15}{95} \cdot \frac{1}{4} \approx 0.0673 \tag{3}$$

(b) We will consider the complement event, that neither of them are green. Let's call the event that at least one of them is green B, this makes the complement $\overline{B}$ the event that neither Skittles are green. Using the same approach as the previous part, we will get the following:

$$\mathbb{P}(\overline{B}) = \mathbb{P}(\overline{B} \cap A) + \mathbb{P}(\overline{B} \cap H) + \mathbb{P}(\overline{B} \cap F) \tag{4}$$
$$= \mathbb{P}(\overline{B} \mid A)\mathbb{P}(A) + \mathbb{P}(\overline{B} \mid H)\mathbb{P}(H) + \mathbb{P}(\overline{B} \mid F)\mathbb{P}(F) \tag{5}$$
$$= 1 \cdot \frac{1}{2} + \frac{80}{90} \cdot \frac{79}{89} \cdot \frac{1}{4} + \frac{80}{95} \cdot \frac{79}{94} \cdot \frac{1}{4} \approx 0.874 \tag{6}$$

This makes our final answer the following:

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) \approx 0.126$$

(c) Let's call the event of having 3 green Skittles G_3. This event is impossible if our roommate ate all the green Skittles.

If they ate half, we have the probability of G_3 as

$$\frac{10 \times 9 \times 8}{90 \times 89 \times 88}.$$

We can see this by noticing that given our roommate ate half the green Skittles, there will be 10 green Skittles left out of the 90 that are still in the bag. After the first one is removed, there will be 9 out of 89 that are green, and so on.

Similarly, if they ate only five green Skittles, we have the probability of G_3 as

$$\frac{15 \times 14 \times 13}{95 \times 94 \times 93},$$

giving us the final result as:

$$\mathbb{P}(G_3) = \mathbb{P}(G_3 \mid A)\mathbb{P}(A) + \mathbb{P}(G_3 \mid H)\mathbb{P}(H) + \mathbb{P}(G_3 \mid F)\mathbb{P}(F) \tag{7}$$
$$= 0 \cdot \frac{1}{2} + \frac{10 \times 9 \times 8}{90 \times 89 \times 88} \cdot \frac{1}{4} + \frac{15 \times 14 \times 13}{95 \times 94 \times 93} \cdot \frac{1}{4} \tag{8}$$
$$\approx 0.00108 \tag{9}$$

(d) We can use the Bayes Rule to solve this.

$$\mathbb{P}(A \mid G_3) = \frac{\mathbb{P}(G_3 \cap A)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid A)\mathbb{P}(A)}{\mathbb{P}(G_3)} = \frac{0 \times 1/2}{0.00108} = 0$$

This makes intuitive sense, since if you took three green Skittles out of the bag, it is impossible that your roommate ate all of them. Using it for the two other conditions, we get:

$$\mathbb{P}(H \mid G_3) = \frac{\mathbb{P}(G_3 \cap H)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid H)\mathbb{P}(H)}{\mathbb{P}(G_3)} = \frac{10 \times 9 \times 8}{90 \times 89 \times 88} \cdot \frac{1}{4} \cdot \frac{1}{0.00108} \approx 0.237$$

$$\mathbb{P}(F \mid G_3) = \frac{\mathbb{P}(G_3 \cap F)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid F)\mathbb{P}(F)}{\mathbb{P}(G_3)} = \frac{15 \times 14 \times 13}{95 \times 94 \times 93} \cdot \frac{1}{4} \cdot \frac{1}{0.00108} \approx 0.763$$

Note that the sum of these probabilities add up to 1.

(e) We can divide this into two cases. If the color of all the Skittles is green, we have already calculated the probability in the previous part.

For all other colors, we can notice that the probabilities will have the same structure, and since these are disjoint events, we can add them to get our final result. Let's find the probability for the case of getting three red Skittles, let's call this event R_3. We find this probability as follows:

$$\begin{aligned} \mathbb{P}(R_3) &= \mathbb{P}(R_3 \mid A)\mathbb{P}(A) + \mathbb{P}(R_3 \mid H)\mathbb{P}(H) + \mathbb{P}(R_3 \mid F)\mathbb{P}(F) && (10) \\ &= \frac{20 \times 19 \times 18}{80 \times 79 \times 78} \cdot \frac{1}{2} + \frac{20 \times 19 \times 18}{90 \times 89 \times 88} \cdot \frac{1}{4} + \frac{20 \times 19 \times 18}{95 \times 94 \times 93} \cdot \frac{1}{4} && (11) \\ &\approx 0.0114 && (12) \end{aligned}$$

If we call the probability of getting three Skittles of the same color X_3, we can find it by adding the probability for the events for different colors such as G_3, and R_3. The probability for getting 3 of the same color for yellow, orange, and purple will be the same as it was for red. Using the same name convention for red and green for the other colors, this can be summed up as:

$$\begin{aligned} \mathbb{P}(X_3) &= \mathbb{P}(G_3) + \mathbb{P}(R_3) + \mathbb{P}(Y_3) + \mathbb{P}(O_3) + \mathbb{P}(P_3) && (13) \\ &= \mathbb{P}(G_3) + 4\mathbb{P}(R_3) && (14) \end{aligned}$$

The above holds since these are all disjoint events, we can't get all three Skittles to be the same color for different colors at the same time. Overall, getting this means we are adding these probabilities, giving us:

$$\mathbb{P}(X_3) = \mathbb{P}(G_3) + 4 \cdot \mathbb{P}(R_3) \approx 0.0468$$

2 Probability Potpourri

Prove a brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) If $|\Omega| = n$, how many distinct events does the probability space have?

(c) Find some probability space Ω and three events A, B, and $C \subseteq \Omega$ such that $\mathbb{P}(A) > \mathbb{P}(B)$ and $\mathbb{P}(A \mid C) < \mathbb{P}(B \mid C)$.

(d) If two events C and D are disjoint and $\mathbb{P}(C) > 0$ and $\mathbb{P}(D) > 0$, can C and D be independent? If so, provide an example. If not, why not?

(e) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(f) Two six sided dice are rolled. Find three events such that they are all pairwise independent, but aren't mutually independent.

Solution:

(a) Start with the right side:

$$\begin{aligned}\mathbb{P}(A) - \mathbb{P}(B) &= [\mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B)] - [\mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)] \\ &= \mathbb{P}(A \setminus B) - \mathbb{P}(B \setminus A) \\ &\leq \mathbb{P}(A \setminus B)\end{aligned}$$

(b) An event is a subset of Ω, and for each outcome, there are 2 options: the outcome is in the event, or it isn't. Since there are n outcomes, there are 2^n events.

(c) Let $\Omega = \{1,2,3\}$, with each outcome being equally likely. Let $A = \{1,2\}$, $B = \{3\}$, and $C = \{3\}$. Then $\mathbb{P}(A) = 2/3 > \mathbb{P}(B) = 1/3$. But $\mathbb{P}(A \mid C) = 0 < \mathbb{P}(B \mid C) = 1$.

(d) No, C and D cannot be independent. If they are independent, then $\mathbb{P}(C \cap D) = \mathbb{P}(C)\mathbb{P}(D) > 0$, but we know that $\mathbb{P}(C \cap D) = 0$ because the events are disjoint.

(e) Using total probability rule:

$$\mathbb{P}(D) = \mathbb{P}(D \cap C) + \mathbb{P}(D \cap \overline{C}) = \mathbb{P}(D \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(D \mid \overline{C}) \cdot \mathbb{P}(\overline{C})$$

But we know that $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, so this simplifies to

$$\mathbb{P}(D) = \mathbb{P}(D \mid C) \cdot [\mathbb{P}(C) + \mathbb{P}(\overline{C})] = \mathbb{P}(D \mid C) \cdot 1 = \mathbb{P}(D \mid C)$$

which defines independence.

(f) Let X be the event that the first die is even, let Y be the event that the second die is even, and let Z be the event that the sum of the two dice is even. Then all three events are pairwise independent, but they are not mutually independent.

3 Identity Theft

A group of n friends go to the gym together, and while they are playing basketball, they leave their bags against the nearby wall. An evildoer comes, takes the student ID cards from the bags, randomly rearranges them, and places them back in the bags, one ID card per bag. What is the probability that no one receives his or her own ID card back? [*Hint*: Use the generalized inclusion-exclusion principle.] Then, find an approximation for the probability as $n \to \infty$.

Solution:

We are looking for the probability of the event that no one receives his or her own ID card back. It is easier to consider the complement of the above event, which is the event that at least one person receives his or her ID card back. Let A_i, $i = 1, \dots, n$, be the event that the ith friend receives his or her own ID card back, so the event we are considering now is $A_1 \cup \cdots \cup A_n$. We will compute this probability using the generalized inclusion-exclusion formula.

- First, we add $\mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)$. Here, $\mathbb{P}(A_i)$ is the probability that the ith friend receives his or her own ID card back, which is $1/n$. So, we add $n \cdot (1/n) = 1$.

- Next, we subtract $\sum_{(i,j)} \mathbb{P}(A_i \cap A_j)$, where the sum runs over all $(i, j) \in \{1, \dots, n\}^2$ with $i < j$. Note that $\mathbb{P}(A_i \cap A_j)$ is the probability that both friend i and friend j receive their own ID cards back, which has probability $(n-2)!/n!$. (To see this, observe that once we have decided that friends i and j will receive their own ID cards back, there are $(n-2)!$ ways to permute the ID cards of the $n-2$ other friends, and there are $n!$ total permutations of the n ID cards.) So, we subtract $\sum_{(i,j)} (n-2)!/n!$, but the summation has $\binom{n}{2}$ terms, so we subtract a total of
$$\binom{n}{2} \frac{(n-2)!}{n!} = \frac{n!}{2!(n-2)!} \cdot \frac{(n-2)!}{n!} = \frac{1}{2!}.$$

- At the kth step of the inclusion-exclusion process, we add $(-1)^{k+1} \sum_{(i_1,\dots,i_k)} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$, where the k-tuples in the summation range over all $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$ with $i_1 < \cdots < i_k$. To compute $\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k})$, note that we have decided that k friends will receive their own ID cards back, the remaining $n-k$ ID cards can be permuted in $(n-k)!$ ways, and there are $n!$ total permutations, so the probability is $(n-k)!/n!$. The summation has a total of $\binom{n}{k}$ terms, so we add
$$(-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = (-1)^{k+1} \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{n!} = (-1)^{k+1} \frac{1}{k!}.$$

Now, adding up all of these probabilities together, we have
$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}.$$
Recall that $A_1 \cup \cdots \cup A_n$ is the *complement* of the event we were originally interested in. So,
$$\mathbb{P}(\text{no friends receive their own ID cards back}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Recall the power series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Therefore, we have the approximation (which gets better as $n \to \infty$):

$$\mathbb{P}(\text{no friends receive their own ID cards back}) \approx \frac{1}{e} \approx 0.368.$$

4 Cookie Jars

You have two jars of cookies, jar 1 and jar 2. Each jar starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$). One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X denote the number of remaining cookies in the two jars. What is the distribution of X?

Solution:

By symmetry, the probability that you found jar 1 empty is the same as the probability that you found jar 2 empty. So, assume that you found jar 1 empty. The probability that $X = k$ and you found jar 1 empty is computed as follows. In order for there to be k cookies remaining, you must have eaten a cookie for $2n-k$ days, and then you must have chosen jar 1 (to discover that it is empty). Within those $2n-k$ days, exactly n of those days you chose jar 1. The probability of this is $\binom{2n-k}{n}2^{-(2n-k)}$. Furthermore, the probability that you then discover jar 1 is empty is $1/2$. So, the probability that $X = k$ and you discover jar 1 empty is $\binom{2n-k}{n}2^{-(2n-k+1)}$. However, we assumed that we discovered jar 1 to be empty; the probability that $X = k$ and jar 2 is empty is the same, so the overall probability that $X = k$ is:

$$\mathbb{P}(X = k) = \binom{2n-k}{n}\frac{1}{2^{2n-k}}, \qquad k \in \{0, \dots, n\}.$$

5 Exploring the Geometric Distribution

In this question, we will further investigate the geometric distribution. Let X, Y be i.i.d. geometric random variables with parameter p. Let $U = \min\{X,Y\}$ and $V = \max\{X,Y\} - \min\{X,Y\}$. Compute the joint distribution of (U,V) and prove that U and V are independent. [*Hint*: If $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ are independent, then $\min\{X,Y\} \sim \text{Geometric}(p+q-pq)$.]

Solution:

One has, for $u,v \in \mathbb{N}$, $u,v \geq 1$:

$$\begin{aligned}
\mathbb{P}(U=u,V=v) &= \mathbb{P}(\min\{X,Y\}=u, \max\{X,Y\}=u+v) \\
&= \mathbb{P}(X=u,Y=u+v)+\mathbb{P}(X=u+v,Y=u) \\
&= \mathbb{P}(X=u)\mathbb{P}(Y=u+v)+\mathbb{P}(X=u+v)\mathbb{P}(Y=u) \\
&= p(1-p)^{u-1}p(1-p)^{u+v-1}+p(1-p)^{u+v-1}p(1-p)^{u-1} = 2p^2(1-p)^{2u+v-2}.
\end{aligned}$$

Also, for $u \in \mathbb{N}$, $u \geq 1$:

$$\begin{aligned}
\mathbb{P}(U=u,V=0) = \mathbb{P}(X=Y=u) = \mathbb{P}(X=u)\mathbb{P}(Y=u) &= p(1-p)^{u-1}p(1-p)^{u-1} \\
&= p^2(1-p)^{2u-2}.
\end{aligned}$$

Putting it together, we have:

$$\mathbb{P}(U=u,V=v) = \begin{cases} 2p^2(1-p)^{2u+v-2} & u,v \in \mathbb{N}, u \geq 1, v \geq 1 \\ p^2(1-p)^{2u-2} & u \in \mathbb{N}, u \geq 1, v=0 \\ 0 & \text{otherwise} \end{cases}$$

Now, to show that U and V are independent, we must compute their marginal distributions. Note that $U=\min\{X,Y\} \sim \text{Geometric}(2p-p^2)$, so

$$\mathbb{P}(U=u) = p(2-p)(1-p)^{2u-2}, \qquad u \in \mathbb{N},\ u \geq 1.$$

(We are using the fact that the minimum of two independent geometric random variables is also geometric.) On the other hand, how do we compute the distribution of V? If $v \in \mathbb{N}$, $v \geq 1$:

$$\begin{aligned}
\mathbb{P}(V=v) &= \sum_{k=1}^{\infty}(\mathbb{P}(X=k,Y=k+v)+\mathbb{P}(X=k+v,Y=k)) \\
&= \sum_{k=1}^{\infty}(\mathbb{P}(X=k)\mathbb{P}(Y=k+v)+\mathbb{P}(X=k+v)\mathbb{P}(Y=k)) \\
&= \sum_{k=1}^{\infty}(p(1-p)^{k-1}p(1-p)^{k+v-1}+p(1-p)^{k+v-1}p(1-p)^{k-1}) \\
&= 2p^2(1-p)^{v-2}\sum_{k=1}^{\infty}((1-p)^2)^k = 2p^2(1-p)^{v-2}\cdot\frac{(1-p)^2}{1-(1-p)^2} = \frac{2p(1-p)^v}{2-p}.
\end{aligned}$$

Otherwise, if $v=0$:

$$\begin{aligned}
\mathbb{P}(V=0) &= \sum_{k=1}^{\infty}\mathbb{P}(X=Y=k) = \sum_{k=1}^{\infty}\mathbb{P}(X=k)\mathbb{P}(Y=k) = \sum_{k=1}^{\infty}p(1-p)^{k-1}p(1-p)^{k-1} \\
&= p^2(1-p)^{-2}\sum_{k=1}^{\infty}((1-p)^2)^k = p^2(1-p)^{-2}\cdot\frac{(1-p)^2}{1-(1-p)^2} = \frac{p}{2-p}.
\end{aligned}$$

It is easily verified that

$$\mathbb{P}(U=u,V=v) = \mathbb{P}(U=u)\mathbb{P}(V=v) \qquad \forall u,v \in \mathbb{R},$$

so U and V are independent.

6 Poisson Coupling

Consider the following discrete joint distribution for $p \in [0,1]$.

$$\begin{aligned}
&\mathbb{P}(X=0,Y=0) = 1-p, \\
&\mathbb{P}(X=1,Y=y) = \frac{e^{-p}p^y}{y!}, && y = 1,2,\ldots, \\
&\mathbb{P}(X=1,Y=0) = e^{-p}-(1-p), \\
&\mathbb{P}(X=x,Y=y) = 0, && \text{otherwise.}
\end{aligned}$$

(a) Recall that all valid distributions satisfy two important properties. Argue that this distribution is a valid joint distribution.

(b) Show that X has the Bernoulli distribution with probability p.

(c) Show that Y has the Poisson distribution with parameter $\lambda = p$.

(d) Show that $\mathbb{P}(X \neq Y) \leq p^2$.

Now, let X_i, $i = 1,2,\ldots$ be a sequence of Bernoulli random variables with probabilities p_i, $i = 1,2,\ldots$. Similarly, let Y_i be a Poisson random variable with parameter $\lambda = p_i$, $i = 1,2,\ldots$. The X_i and Y_i are coupled, so that they have the joint distribution described above (with $p = p_i$), but for $i \neq j$, (X_i,Y_i) and (X_j,Y_j) are independent.

We will now introduce a coupling argument which shows that the distribution of $\sum_{i=1}^n X_i$ approaches a Poisson distribution with parameter $\lambda = p_1 + \cdots + p_n$.

(e) A common way to measure the "distance" between two probability distributions is known as the total variation norm, and it is given by

$$d(X,Y) = \frac{1}{2}\sum_{k=0}^{\infty} |\mathbb{P}(X=k) - \mathbb{P}(Y=k)|.$$

Show that $d(X,Y) \leq \mathbb{P}(X \neq Y)$. [*Hint*: Use the Law of Total Probability to split up the events according to $\{X=Y\}$ and $\{X \neq Y\}$.]

(f) Show that $\mathbb{P}(\sum_{i=1}^n X_i \neq \sum_{i=1}^n Y_i) \leq \sum_{i=1}^n \mathbb{P}(X_i \neq Y_i)$. [*Hint*: Maybe try the Union Bound.]

(g) Finally, for the X_i and Y_i defined above, show that $d(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) \leq \sum_{i=1}^n p_i^2$.

Solution:

(a) We need to verify that the probabilities sum to 1. Indeed,

$$\begin{aligned}
\mathbb{P}(X=0,Y=0) + \mathbb{P}(X=1,Y=0) + \sum_{y=1}^{\infty}\mathbb{P}(X=1,Y=y) &= e^{-p} + \sum_{y=1}^{\infty}\frac{e^{-p}p^y}{y!} \\
&= \cancel{e^{-p}} + 1 - \cancel{e^{-p}} = 1.
\end{aligned}$$

Also, the probabilities are non-negative, since $e^{-p} \geq 1-p$ always.

(b) We know that $\mathbb{P}(X=0)=\mathbb{P}(X=0,Y=0)=1-p$, and that $\mathbb{P}(X=x)=0$ for any $x \notin \{0,1\}$. Then $\mathbb{P}(X=0)+\mathbb{P}(X=1)=1$, so $\mathbb{P}(X=1)=p$. This is a sufficient approach, but to be fully explicit, we can verify through direct calculation that $\mathbb{P}(X=1)=p$:

$$\mathbb{P}(X=1)=\mathbb{P}(X=1,Y=0)+\sum_{y=1}^{\infty}\mathbb{P}(X=1,Y=y)=e^{-p}-(1-p)+\sum_{y=1}^{\infty}\frac{e^{-p}p^y}{y!}$$
$$=\cancel{e^{-p}}-1+p+\cancel{1-e^{-p}}=p.$$

Hence, X has the Bernoulli distribution with probability of success p.

(c) We see that $\mathbb{P}(Y=0)=\mathbb{P}(X=0,Y=0)+\mathbb{P}(X=1,Y=0)=e^{-p}$, and for $y=1,2,\ldots$ we have

$$\mathbb{P}(Y=y)=\mathbb{P}(X=1,Y=y)=\frac{e^{-p}p^y}{y!}.$$

This is indeed the Poisson distribution with rate $\lambda=p$.

(d) We can recognize that $\mathbb{P}(X\neq Y)=1-\mathbb{P}(X=Y)$:

$$\begin{aligned}\mathbb{P}(X\neq Y)=1-\mathbb{P}(X=Y)&=1-\mathbb{P}(X=0,Y=0)-\mathbb{P}(X=1,Y=1)\\&=1-(1-p)-\frac{e^{-p}p^1}{1!}\\&=p-e^{-p}p\\&=p(1-e^{-p})\leq p^2.\end{aligned}$$

In the last line, we are using $1-e^{-p}\leq p$. Note that this follows from $e^{-p}\geq 1-p$ by rearranging the inequality.

Alternatively, we can compute $\mathbb{P}(X\neq Y)$ directly:

$$\mathbb{P}(X\neq Y)=\mathbb{P}(X=1,Y=0)+\mathbb{P}(X=1,Y\geq 2)=e^{-p}-(1-p)+\sum_{y=2}^{\infty}\frac{e^{-p}p^y}{y!}$$
$$=e^{-p}-(1-p)+1-e^{-p}-pe^{-p}=p(1-e^{-p})\leq p^2.$$

(e) One has

$$\begin{aligned}d(X,Y)&=\frac{1}{2}\sum_{k=0}^{\infty}|\mathbb{P}(X=k)-\mathbb{P}(Y=k)|\\&=\frac{1}{2}\sum_{k=0}^{\infty}|\mathbb{P}(X=k,X=Y)+\mathbb{P}(X=k,X\neq Y)-\mathbb{P}(Y=k,X=Y)\\&\qquad\qquad-\mathbb{P}(Y=k,X\neq Y)|.\end{aligned}$$

Note that the event $\{X=k, X=Y\}$ is the same as $\{Y=k, X=Y\}$ (they both equal the event $\{X=Y=k\}$). Hence, these terms cancel and we have

$$\begin{aligned} d(X,Y) &= \frac{1}{2}\sum_{k=0}^{\infty} |\mathbb{P}(X=k, X\neq Y) - \mathbb{P}(Y=k, X\neq Y)| \\ &\leq \frac{1}{2}\left(\sum_{k=0}^{\infty} \mathbb{P}(X=k, X\neq Y) + \sum_{k=0}^{\infty} \mathbb{P}(Y=k, X\neq Y)\right) = \frac{1}{2}\left(\mathbb{P}(X\neq Y) + \mathbb{P}(X\neq Y)\right). \end{aligned}$$

We see that the factor of $1/2$ disappears and we are left with

$$d(X,Y) \leq \mathbb{P}(X\neq Y). \tag{15}$$

(f) Note that the event $\{\sum_{i=1}^n X_i \neq \sum_{i=1}^n Y_i\} \subseteq \{\exists i\ X_i \neq Y_i\}$, since if the two summations $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ are different, then there must be at least one term which is different between the summations. Now, we can write

$$\mathbb{P}\left(\sum_{i=1}^n X_i \neq \sum_{i=1}^n Y_i\right) \leq \mathbb{P}(X_i \neq Y_i \text{ for some } i) = \mathbb{P}\left(\bigcup_{i=1}^n \{X_i \neq Y_i\}\right).$$

Now, we apply the Union Bound to the term on the right to obtain

$$\mathbb{P}\left(\sum_{i=1}^n X_i \neq \sum_{i=1}^n Y_i\right) \leq \sum_{i=1}^n \mathbb{P}(X_i \neq Y_i). \tag{16}$$

(g) Thanks to the inequalities we have proved, we can write down

$$d\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right) \underbrace{\leq}_{(15)} \mathbb{P}\left(\sum_{i=1}^n X_i \neq \sum_{i=1}^n Y_i\right) \underbrace{\leq}_{(16)} \sum_{i=1}^n \mathbb{P}(X_i \neq Y_i) \leq \sum_{i=1}^n p_i^2.$$

The last inequality is from part (d).

This is known as Le Cam's Theorem. It provides precise bounds on how far the sum of independent Bernoulli random variables is from a Poisson distribution.

7 Joint Distributions

(a) Suppose that X_i, $i = 1, \ldots, n$ are binary-valued random variables. How many parameters are required to parameterize the joint distribution $\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$?

(b) Continuing from the previous part, suppose that all X_is are independent. How many parameters are required to parameterize the joint distribution?

Solution:

(a) There are $2^n - 1$ parameters required. There are 2^n entries in the joint distribution, but the values must sum to 1, so there are only $2^n - 1$ free parameters.

(b) Each random variable requires only 1 parameter, so in total only n parameters are required.

8 Indicator Variables

(a) After throwing n balls into m bins at random, what is the expected number of bins that contains exactly k balls?

(b) Alice and Bob each draw k cards out of a deck of 52 distinct cards with replacement. Find k such that the expected number of common cards that both Alice and Bob draw is at least 1.

(c) How many people do you need in a room so that you expect that there is going to be a shared birthday on a Monday of the year (assume 52 Mondays in a year and 365 days in a year)?

Solution:

(a) Let $X_i = 1$ if bin i contains exactly k balls and $X_i = 0$ otherwise.

$$\mathbb{E}[X_i] = \binom{n}{k}\left(\frac{1}{m}\right)^k\left(\frac{m-1}{m}\right)^{n-k} = \binom{n}{k}\frac{(m-1)^{n-k}}{m^n}$$

$$\mathbb{E}[X] = \sum_{i=1}^{m}\binom{n}{k}\frac{(m-1)^{n-k}}{m^n} = \binom{n}{k}\frac{(m-1)^{n-k}}{m^{n-1}}$$

(b) Let $X_i = 1$ if card i is chosen by both Alice and Bob and $X_i = 0$ otherwise.
After drawing k cards, the probability of drawing a particular card is $1-(51/52)^k$ so

$$\mathbb{E}[X_i] = \left(1-\left(\frac{51}{52}\right)^k\right)\cdot\left(1-\left(\frac{51}{52}\right)^k\right)$$

$$\mathbb{E}[X] = \sum_{i=1}^{52}\left(1-\left(\frac{51}{52}\right)^k\right)^2 = 52\cdot\left(1-\left(\frac{51}{52}\right)^k\right)^2.$$

Setting $\mathbb{E}[X] = 1$, we have $k = 7.69 \approx 8$.

(c) For $i < j$, let $X_{i,j} = 1$ if i, j share a birthday and $X_{i,j} = 0$ otherwise. Then, the total number of shared birthdays is $X = \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} X_{i,j}$, where k is the total number of people in the room. There is $52/365$ chance that person i has a birthday on a Monday and $1/365$ chance that person j has same birthday as person i so

$$\mathbb{E}[X] = \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\frac{52}{365}\cdot\frac{1}{365} = \binom{k}{2}\frac{52}{365^2} = \frac{k(k-1)}{2}\cdot\frac{52}{365^2}.$$

We want $\mathbb{E}[X] = 1$ so $k \approx 72$.