CS 70 Discrete Mathematics and Probability Theory
Fall 2020

HW 5

Due: Friday 10/02 at 10:00 PM
Grace period until Friday 10/02 at 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 RSA with Just One Prime

Given the message $x \in \{0,1,\ldots,N-1\}$ and $N = pq$, where p and q are prime numbers, conventional RSA encrypts x with $y = E(x) \equiv x^e \pmod{N}$. The decryption is done by $D(y) \equiv y^d \pmod{N}$, where d is the inverse of $e \pmod{(p-1)(q-1)}$.

Alice is trying to send a message to Bob, and as usual, Eve is trying to decipher what the message is. One day, Bob gets lazy and tells Alice that he will now use $N = p$, where p is a 1024-bit prime number, as part of his public key. He tells Alice that it's okay, since Eve will have to try out 2^{1024} combinations to guess x. It is very likely that Eve will not find out the secret message in a reasonable amount of time! In this problem, we will see whether Bob is right or wrong. Assume that Eve has found out about this new setup and that she knows the public key.

Similar to the original method, for any message $x \in \{0,1,\ldots,N-1\}$, $E(x) \equiv x^e \pmod{p}$, and $D(y) \equiv y^d \pmod{p}$. Choose e such that it is coprime with $p-1$, and choose $d \equiv e^{-1} \pmod{p-1}$.

(a) Prove that the message x is recovered after it goes through your new encryption and decryption functions, $E(x)$ and $D(y)$.

(b) Can Eve compute d in the decryption function? If so, by what algorithm and approximately how many iterations does it take for it to terminate?

(c) Given part (b), how would Eve recover x and what algorithm would she use? Approximately how many iterations does it take to terminate?

(d) Based on the previous parts, can Eve recover the original message in a reasonable amount of time? Explain.

2 Squared RSA

(a) Prove the identity $a^{p(p-1)} \equiv 1 \pmod{p^2}$, where a is coprime to p, and p is prime. (Hint: Try to mimic the proof of Fermat's Little Theorem from the notes.)

(b) Now consider the RSA scheme: the public key is $(N = p^2q^2, e)$ for primes p and q, with e relatively prime to $p(p-1)q(q-1)$. The private key is $d = e^{-1} \pmod{p(p-1)q(q-1)}$. Prove that the scheme is correct for x relatively prime to both p and q, i.e. $x^{ed} \equiv x \pmod N$. (Hint: Try to mimic the proof of RSA correctness from the notes.)

3 The CRT and Lagrange Interpolation

Let $n_1, \dots n_k$ be pairwise co-prime, i.e. n_i and n_j are co-prime for all $i \neq j$. The Chinese Remainder Theorem (CRT) tells us that there exist solutions to the following system of congruences:

$$x \equiv a_1 \pmod{n_1} \quad (1)$$
$$x \equiv a_2 \pmod{n_2} \quad (2)$$
$$\vdots \quad (\vdots)$$
$$x \equiv a_k \pmod{n_k} \quad (k)$$

and all solutions are equivalent $\pmod{n_1 n_2 \cdots n_k}$. For this problem, parts (a)-(c) will walk us through a proof of the Chinese Remainder Theorem. We will then use the CRT to revisit Lagrange interpolation.

(a) We start by proving the $k = 2$ case: Prove that we can always find an integer x_1 that solves (1) and (2) with $a_1 = 1, a_2 = 0$. Similarly, prove that we can always find an integer x_2 that solves (1) and (2) with $a_1 = 0, a_2 = 1$.

(b) Use part (a) to prove that we can always find at least one solution to (1) and (2) for any a_1, a_2. Furthermore, prove that all possible solutions are equivalent $\pmod{n_1 n_2}$.

(c) Now we can tackle the case of arbitrary k: Use part (b) to prove that there exists a solution x to (1)-(k) and that this solution is unique $\pmod{n_1 n_2 \cdots n_k}$.

For polynomials $p_1(x)$, $p_2(x)$ and $q(x)$ we say that $p_1(x) \equiv p_2(x) \bmod q(x)$ if $p_1(x) - p_2(x)$ is of the form $q(x) \times m(x)$ for some polynomial $m(x)$.

(d) Define the polynomials $x - a$ and $x - b$ to be co-prime if they have no common divisor of degree 1. Assuming that the CRT still holds when replacing x, a_i and n_i with polynomials (using the definition of co-prime polynomials just given), show that the system of congruences

$$p(x) \equiv y_1 \pmod{(x - x_1)} \quad (1')$$
$$p(x) \equiv y_2 \pmod{(x - x_2)} \quad (2')$$
$$\vdots \quad (\vdots)$$
$$p(x) \equiv y_k \pmod{(x - x_k)} \quad (k')$$

has a unique solution $\pmod{(x-x_1)\cdots(x-x_k)}$ whenever the x_i are pairwise distinct. What is the connection to Lagrange interpolation?

Hint: To show that a unique solution exists, you may use the fact that the CRT has a unique solution when certain properties are satisfied.

4 Polynomials in Fields

Define the sequence of polynomials by $P_0(x) = x+12$, $P_1(x) = x^2-5x+5$ and $P_n(x) = xP_{n-2}(x) - P_{n-1}(x)$.

(For instance, $P_2(x) = 17x-5$ and $P_3(x) = x^3-5x^2-12x+5$.)

(a) Show that $P_n(7) \equiv 0 \pmod{19}$ for every $n \in \mathbb{N}$.

(b) Show that, for every prime q, if $P_{2017}(x) \not\equiv 0 \pmod{q}$, then $P_{2017}(x)$ has at most 2017 roots modulo q.

5 Secrets in the United Nations

A vault in the United Nations can be opened with a secret combination $s \in \mathbb{Z}$. In only two situations should this vault be opened: (i) all 193 member countries must agree, or (ii) at least 55 countries, plus the U.N. Secretary-General, must agree.

(a) Propose a scheme that gives private information to the Secretary-General and all 193 member countries so that the secret combination s can only be recovered under either one of the two specified conditions.

(b) The General Assembly of the UN decides to add an extra level of security: each of the 193 member countries has a delegation of 12 representatives, all of whom must agree in order for that country to help open the vault. Propose a scheme that adds this new feature. The scheme should give private information to the Secretary-General and to each representative of each country.

6 Secret Sharing with Spies

An officer stored an important letter in her safe. In case she is killed in battle, she decides to share the password (which is a number) with her troops. However, everyone knows that there are 3 spies among the troops, but no one knows who they are except for the three spies themselves. The 3 spies can coordinate with each other and they will either lie and make people not able to open the safe, or will open the safe themselves if they can. Therefore, the officer would like a scheme to share the password that satisfies the following conditions:

- When M of them get together, they are guaranteed to be able to open the safe even if they have spies among them.

- The 3 spies must not be able to open the safe all by themselves.

Please help the officer to design a scheme to share her password. What is the scheme? What is the smallest M? Show your work and argue why your scheme works and any smaller M couldn't work. (The troops only have one chance to open the safe; if they fail the safe will self-destruct.)