CS 70 Discrete Mathematics and Probability Theory
Fall 2020 HW 7

1 Strings

How many different strings only contains A,B,C? And how many such strings contains at least one of each characters?

Solution: 3^5 since each position have 3 different choices.

Let E_A be the event that character A doesn't exists in the string, similar for E_B, E_C. Then the total number of bad event is $|E_A + E_B + E_C|$

By the Principle of Inclusion and Exclusion,

$$|E_A+E_B+E_C| = |E_A|+|E_B|+|E_C|-|E_A\cap E_B|-|E_A\cap E_C|-|E_B\cap E_C|+|E_A\cap E_B\cap E_C| = 3*2^5-3*1 = 93$$

, so the total number of valid string is $3^5 - 93 = 150$

2 Palindromes

How many 5-digit palindromes are there? (A palindrome is a number that reads the same way forwards and backwards. For example, 27872 and 48484 are palindromes, but 28389 and 12541 are not.)

Solution:

We construct the number from left-to-right. We have 9 choices for the first digit (since it can't be 0), then 10 choices for the second digit, then 10 choices for the third digit. But now we're out of choices: the fourth digit must match the second, and the last digit must match the first. Therefore, there are $9\cdot 10\cdot 10 = 900$ such numbers.

3 Maze in general and Trees too!!!

Given an maze of sidelength n where one starts at $(0,0)$ and goes to (n,n).

(a) How many shortest paths are there that go from $(0,0)$ to (n,n)?

(b) Extending the width by 1, how many shortest paths are there that go from $(0,0)$ to $(n-1,n+1)$.

(c) Now consider shortest paths that meet the conditions which only use to points (x,y) where $y \le x$. That is, the path cannot cross line $y = x$.

   i. Give an expression using part (a) and (b), that counts the number of paths. (Hint: consider what happens after a shortest that crosses $y = x$ at (i,i), that is, the remaining path starting from $(i,i+1)$ and then continuing to (n,n). If in the remainder of the path, one exchanges the y-direction moves with x-direction moves and vice versa, where does one end up?

ii. A different tack is to derive a recursive formula. We call these paths n-legal paths for a maze of sidelength n, and let F_n be the number of n-legal paths.

Consider a path, and let $i < n$ be the largest value where the path contains (i,i), argue the number of paths is then $F_i * F_{n-i-1}$.

(Hint: if $i = 0$, what are your first and last moves, and where is the remainder of the path allowed to go.)

iii. Give a recursive formula for the number of spanning trees of a complete graph K_n for $n \geq 3$, where each non-root node has degree 3 or 1, and at most 1 node has degree 2?

Two trees are different if and only if either left-subtree is different or right-subtree is different.

(Notice something about your formula and the maze problem. Neat!)

Solution: Let $(x,y) \rightarrow (x+1,y)$ be a move 'right' command. And $(x,y) \rightarrow (x,y+1)$ be a move 'up' command.

(a) It's $\binom{2n}{n}$ as there are total number of $2n$ moves and n of them are move 'up' command, the rest of them are move 'right' command.

(b) It's $\binom{2n}{n-1}$ as there are $n-1$ move 'right' command.

- The solution is to count the number of pathes that cross $y = x$. Once a path crosses $y = x$, we flip the later protion of the path. Let the invalid path first time crosses $y = x$ at (i,i) and arrives at $(i,i+1)$. Then if we do not flip the path, it will arrive (n,n) takes $n-i$ "go right" command, and $n-1-i$ "go up" command. If we flip these command, it will go to $(i+n-1-i, i+1+n-i) = (n-1,n+1)$. So all invalid pathes maps to a path from $(0,0)$ to $(n-1,n+1)$. Next we argue that all path from $(0,0)$ to $(n-1,n+1)$ maps to a invalid path:

  Pathes from $(0,0)$ to $(n-1,n+1)$ must cross the line $y = x$, let it first cross the line at (i,i) and arrives $(i,i+1)$. Then it takes $n-1-i$ go right command, and $n+1-(i+1)$ go up command. We flip these command, $n-1-i$ go up command, $n+1-i$ go right command, then the path will arrive $(i+n+1-(i+1), i+1+n-1-i) = (n,n)$ and this new path is considered as invalid path since it crosses $y = x$ at point (i,i). So all $(0,0) -> (n-1,n+1)$ pathes can be mapped into a invalid pathes.

  So there is a bijective mapping between invalid pathes and $(0,0) -> (n-1,n+1)$ pathes. The total number is $\binom{2n}{n} - \binom{2n}{n-1}$

- Let F_n be the total number of different ways from $(0,0)$ to (n,n) satisfies the condition above. We know $F_0 = 1$. Let (i,i) be the last point on line $y = x$ that a path touches except for (n,n). Then total number of such path is $F_i * F_{n-1-i}$ where F_i is the total number of pathes from $(0,0)$ to (i,i). Since (i,i) is the last boundry point it touches, so for all later steps, it must not cross the line $y = x-1$, it's equvalient to say the total number of pathes from $(i+1,i)$ to $(n,n-1)$, it's F_{n-1-i}. F_n is a summation of all these products. $F_n = \sum_{i=0}^{n-1} F_i * F_{n-1-i}$.

- Let T_n be the total number of different trees with n nodes. The number of different trees when the left subtree has size i and right subtree has size $n-i-1$ is $T_i T_{n-i-1}$. If we sum over all possible sizes of left subtrees, we can get the total number of different trees that is structurally different: $T_n = \sum_{i=0}^{n-1} T_i T_{n-i-1}$. And the same counting arguments captures totally different objects! (Maze and trees).