CS 70 Fall 2020 — Discrete Mathematics and Probability Theory — HW 8

1 Countability Proof Practice

(a) A disk is a 2D region of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap. Is this set always countable, or potentially uncountable?
(*Hint*: Attempt to relate it to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$)

(b) A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap. Is this set always countable, or potentially uncountable?
(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)

(c) Is the set containing all increasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \geq f(y)$) countable or uncountable? Prove your answer.

(d) Is the set containing all decreasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \leq f(y)$) countable or uncountable? Prove your answer.

Solution:

(a) Countable. Each disk must contain at least one rational point (an (x,y)-coordinate where $x, y \in \mathbb{Q}$) in its interior, and due to the fact that no two disks overlap, the cardinality of the set of disks can be no larger than the cardinality of $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable.

(b) Possibly uncountable. Consider the circles $C_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ for each $r \in \mathbb{R}$. For $r_1 \neq r_2$, C_{r_1} and C_{r_2} do not overlap, and there are uncountably many of these circles (one for each real number).

(c) Suppose that there is a bijection between $\mathbb{N}$ and the set of all increasing functions $\mathbb{N} \to \mathbb{N}$:

$$\begin{aligned}
0 &\mapsto (f_0(0), f_0(1), f_0(2), \ldots) \\
1 &\mapsto (f_1(0), f_1(1), f_1(2), \ldots) \\
2 &\mapsto (f_2(0), f_2(1), f_2(2), \ldots) \\
&\vdots
\end{aligned}$$

We will use a diagonalization argument to prove that there is a function f which is not in the above list. Define

$$f(n) = 1 + \sum_{i=1}^{n} f_i(n).$$

First, we will show that f is increasing. Indeed, if $m \leq n$, then

$$f(m) = 1 + \sum_{i=1}^{m} f_i(m) \leq 1 + \sum_{i=1}^{n} f_i(m) \leq 1 + \sum_{i=1}^{n} f_i(n) = f(n).$$

The first inequality is because each function is non-negative; the second inequality is because the f_i are increasing.

To show that f is not in the list, note that

$$f(n) = 1 + \sum_{i=1}^{n} f_i(n) \geq 1 + f_n(n) > f_n(n).$$

Since $f(n) > f_n(n)$ for each $n \in \mathbb{N}$, f cannot be any of the functions in the list. Therefore, the set of increasing functions $f : \mathbb{N} \to \mathbb{N}$ is uncountable.

(d) Given any function that begins with $f(0) = n$, consider the number of indices in which the function decreases in output: the set of i such that $f(i) < f(i-1)$. The range of f is a subset of $\mathbb{N}$ so by the well-ordering principle there must be a least element. Call this element a. Then there are only at most $n - a$ transition points. We can set a bijection for any function with $f(0) = n$ to a "word" of indices at which the function decreases. Therefore, the set of decreasing functions $\mathbb{N} \to \mathbb{N}$ has the same cardinality as the set of finite bit strings from a countably infinite alphabet, which is countable. Therefore, the set of all decreasing functions is countable.

2 Hilbert's Hotel

You don't have any summer plans, so you decide to spend a few months working for a magical hotel with a countably infinite number of rooms. The rooms are numbered according to the natural numbers, and all the rooms are currently occupied. Assume that guests don't mind being moved from their current room to a new one, so long as they can get to the new room in a finite amount of time (i.e. guests can't be moved into a room infinitely far from their current one).

(a) A new guest arrives at the hotel. All the current rooms are full, but your manager has told you never to turn away a guest. How could you accommodate the new guest by shuffling other guests around? What if you instead had k guest arrive, for some fixed, positive $k \in \mathbb{Z}$?

(b) Unfortunately, just after you've figured out how to accommodate your first $k+1$ guests, a countably infinite number of guests arrives in town on an infinitely long train. The guests on the train are sitting in seats numbered according to the natural numbers. How could you accommodate all the new guests?

(c) Thanks to a (literally) endless stream of positive TripAdvisor reviews, word of the infinite hotel gets around quickly. Soon enough you find out that a countably infinite number of trains have arrived in town. Each is of infinite length, and carries a countably infinite number of passengers. How would you accommodate all the new passengers?

Solution:

(a) Shift all guests into the room number that is k greater than their current room number. So for a guest in room i move him/her to room $i+k$. Then place the k new guests in the k first rooms in the hotel which will now be unoccupied.

(b) Place all existing guests in room $2i$ where i is their current room number. Place all the new guests in room $2j+1$ where j is their seat number on the train.

(c) **Solution 1**: We first set up a bijection between the newly arriving guests and the set $\mathbb{N}\times\mathbb{N}$. Notice that each guest has an "address": his/her train number i and his/her seat number j. Let this guest be mapped to (i,j). It is clear that this is a bijection.

We know from Lecture Note 10 that the set $\mathbb{N}\times\mathbb{N}$ is countable (via the spiral method) and hence there is a bijection from $\mathbb{N}^2$ to $\mathbb{N}$. Thus the newly arriving guests can be enumerated and considered as if arriving in a single infinite length train with their corresponding seat numbers given by the enumeration. This reduces to the same exact problem as the previous part! Therefore, we can accommodate these guests.

Solution 2: Place all existing guests in room 2^i where i is their current room number. Assign the $(k+2)$th prime, p_{k+2}, to the kth train (e.g. the 0th train will be assigned the 2nd prime, 3). We then place each new guest in room p_{k+2}^{j+1}, where j is the seat number of the new guest on that train.

This works because any power of a prime p will not have any prime factors other than p.

Yes, there will be plenty of empty rooms, but that's okay because every guest will still have somewhere to stay.

3 Finite and Infinite Graphs

The graph material that we learned in lecture still applies if the set of vertices of a graph is infinite. We thus make a distinction between finite and infinite graphs: a graph $G=(V,E)$ is finite if V and E are both finite. Otherwise, the graph is infnite. As examples, consider the graphs

- $G_1 = (V=\mathbb{Z},\ E=\{(i,j)\in\mathbb{Z}\times\mathbb{Z} \mid |i-j|=1\})$
- $G_2 = (V=\mathbb{Z},\ E=\{(i,j)\in\mathbb{Z}\times\mathbb{Z} \mid i<j\})$
- $G_3 = (V=\mathbb{Z}^2,\ E=\{((i,j),(k,l))\in\mathbb{Z}^2\times\mathbb{Z}^2 \mid (i=k\wedge|j-l|=1)\vee(j=l\wedge|i-k|=1)\})$

Observe that G_1 is a line of integers, G_2 is a complete graph over all integers, and G_3 is an grid of integers. Prove whether the following sets of graphs are countable or uncountable

(a) The set of all finite graphs $G=(V,E)$, for $V\subseteq\mathbb{N}$

(b) The set of all infinite graphs over a fixed, countably infinite set of vertices (in other words, they all have the same vertex set).

(c) The set of all graphs over a fixed, countably infinite set of vertices, the degree of each vertex is exactly two. For instance, every vertex in G_1 (defined above) has degree 2.

(d) We say that graphs $G = (V,E)$ and $G' = (V',E')$ are isomorphic if the exists some bijection $f : V \to V'$ such that $(u,v) \in V$ iff $(f(u), f(v)) \in V'$. Such a bijection f is called a **graph isomorphism**. Suppose we consider two graphs to be the equivalent if they are isomorphic. The idea is that if we relabel the vertices of a graph, it is still the same graph. Using this definition of "being the same graph", can you conclude that the set of trees over countably infinite vertices is countable?
(*Hint:* Begin by showing that for any graph isomorphism f, and any vertex v, $f(v)$ and v have the same degree)

Solution:

(a) Countable. Let A be the set of graphs we are counting. Let A_k be the set of all graphs $G = (V,E)$, where $V \subseteq \{1,2,\dots k\}$. A_k is finite because there are only 2^k possible subsets of vertices that is a subset of $\{1,2,...k\}$. For a particular vertex set of size $q \leq k$, there are $\binom{q}{2}$ possible edges over that particular vertex set. Since the number of possible graphs over the vertex set is the power set of all the possible edges to choose from, the number of possible graphs on q vertices is at most $2^{\binom{q}{2}} \leq 2^{k^2}$. There are 2^k ways of choosing a vertex set $V \subseteq \{1,2,\dots,k\}$, so the number of graphs $|A_k|$ of at most k vertices is bounded at most $2^k \cdot 2^{k^2} = 2^{k^2+k}$. Since each graph's vertex set is a subset of $\mathbb{N}$, the graph must be must be contained in A_k for some k. Thus, $A = \cup_{k=1}^{\infty} A_k$. We can simply enumerate A by enumerating each A_k. Note we are double counting some graphs but for it purpose of showing countability, it's okay.

(b) Uncountable. The set of possible edges in a graph of countably infinite vertices is clearly infinite. The power set of any infinte set is uncountable.

(c) Uncountable.

Recall from lectures that the number of infinite-length binary strings is uncountably infinite. We will construct an injection from this set to the given set of graphs.

First observe that since the number of vertices is countable, we can label them with the positive integers: $V = \{v_i | \, i \in \{1,2,3,\dots\}\}$. Now consider an infinite binary string $s = b_1 b_2 b_3 \dots \in \{0,1\}^\infty$, where $b_i \in \{0,1\}$ is the ith digit of the string. We can encode the first digit by creating a simple cycle of length $b_1 + 3$ out of vertices $v_1, v_2, \dots, v_{b_1+2}$ (i.e. we make a chain $v_1 \to v_2 \to \cdots \to v_{b_1+2} \to v_1$). We can also encode b_2 by creating a simple cycle of length $b_2 + 3$ out of the vertices $v_{b_1+3}, v_{b_1+4}, \dots, v_{b_1+b_2+5}$. Proceeding in this way, we encode the ith digit by making a simple cycle of length $b_i + 3$ out of the vertices $v_{f_s(i)}$ through $v_{f_s(i)+b_i+1}$, where $f_s(i) = 3(i-1) + \sum_{j=1}^{i-1} b_j$. Each vertex in V will be part of exactly one such cycle, and so each vertex will have degree exactly two.

Two see that this is an injection, consider two distinct strings $s = b_1 b_2 b_3 \dots$ and $s' = b'_1 b'_2 b'_3 \dots$ that first differ in their ith digit. By construction, the subgraph formed by the first $f_s(i-1)$ vertices will be identical for the each of the graphs. However, the next $b_i + 3$ vertices will be part of a length-(b_i+3) simple cycle in the graph for s, while the next b'_i+3 vertices will be part of a length-(b'_i+3) simple cycle in the graph for s', with $b'_i+3 \neq b_i+3$. This holds for all $s, s' \in \{0,1\}^\infty$, so we have an injection.

(d) Uncountable. First, to show that if f is a graph isomorphism, then v and $f(v)$ has the same degree, suppose that they do not have the same degree. Then either there is some neighbor, w, of v such that $(f(v), f(w))$ is not in E' or there is some neighbor w', of $f(v)$ such that $(v, f^{-1}(w'))$ is not in E. Either way, f is not an isomorphism.

We will inject the set of infinite bit strings into the set of infinite trees. Specifically, we will construct these trees by adding leaves to the infinite line graph $G=(V,E)$, where $V=\{v_0, v_2, \ldots\}$ and $E=\{(v_i, v_{i+1}) \mid i \in \mathbb{N}\}$. For each bit string, b, where $b(i)$ is the i^{th} bit of b, construct G_b as follows: First, add 10 leaves to v_0. Then, for each i, if $b(i)=1$, add a leaf to v_i. We will denote each v_i in G_b as v_i^b. Clearly, the resulting graph is a tree, as adding leaves will never disconnect a graph or create cycles. The graph also has a countable number of vertices because we've only added countably many vertices. To show injection, we first note that any graph isopmorphism must map a vertex to a vertex of the same degree. Now suppose $b \neq b'$ but there is an isomorphism, f, from G_b to $G_{b'}$. By the method of construction each G_b has exactly one vertex of degree ≥ 10. Thus, any isomorphism from the vertex set of G_b to that of $G_{b'}$ must map v_0^b to $v_0^{b'}$. Thus, for each i, it must map v_i^b to $v_i^{b'}$. However, because there is some i where $b(i) \neq b'(i)$, there is some i where where $f(v_i^b)$ and $f(v_i^{b'})$ have different degrees.