

Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 Faulty Lightbulbs

Box 1 contains 1000 lightbulbs of which 10% are defective. Box 2 contains 2000 lightbulbs of which 5% are defective.

- (a) Suppose a box is given to you at random and you randomly select a lightbulb from the box. If that lightbulb is defective, what is the probability you chose Box 1?
- (b) Suppose now that a box is given to you at random and you randomly select two lightbulbs from the box. If both lightbulbs are defective, what is the probability that you chose from Box 1?

2 Solve the Rainbow

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

- (a) If you take a Skittle from the bag, what is the probability that it is green?
- (b) If you take two Skittles from the bag, what is the probability that at least one is green?
- (c) If you take three Skittles from the bag, what is the probability that they are all green?
- (d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?
- (e) If you take three Skittles from the bag, what is the probability that they are all the same color?

3 Easter Eggs

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You're given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs without replacement.

- (a) What is the probability that the first egg you drew was a chocolate egg?
- (b) What is the probability that the second egg you drew was a chocolate egg?
- (c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

4 Cliques in Random Graphs

Consider a graph $G(V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v , we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability $1/2$, $G(V, E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two isolated vertices.

- (a) What is the size of the sample space?
- (b) A k -clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k -clique?
- (c) Prove that the probability that the graph contains a k -clique for $k = 4\lceil \log n \rceil + 1$ is at most $1/n$. You may use the fact that $\binom{n}{k} \leq n^k$ without proof.

5 Identity Theft

A group of n friends go to the gym together, and while they are playing basketball, they leave their bags against the nearby wall. An evildoer comes, takes the student ID cards from the bags,

randomly rearranges them, and places them back in the bags, one ID card per bag. What is the probability that no one receives his or her own ID card back? [*Hint*: Use the generalized inclusion-exclusion principle.]

Then, find an approximation for the probability as $n \rightarrow \infty$. You may, without proof, refer to the power series for e^x :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

6 Playing Strategically

Bob, Eve and Carol bought new slingshots. Bob is not very accurate hitting his target with probability $1/3$. Eve is better, hitting her target with probability $2/3$. Carol never misses. They decide to play the following game: They take turns shooting each other. For the game to be fair, Bob starts first, then Eve and finally Carol. Any player who gets shot has to leave the game. What is Bob's best course of action regarding his first shot?

- (a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first.
- (b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.
- (c) Compute the probability of the same events for a duel of Bob against Carol.
- (d) Assuming that both Eve and Carol play rationally, conclude that Bob's best course of action is to shoot into the air (i.e., intentionally miss)! (Hint: What happens if Bob misses? What if he doesn't?)