CS 70 Spring 2017 — Discrete Mathematics and Probability Theory — Rao — HW 8

1 Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

2 Story Problems

Prove the following identities by combinatorial argument:

(a) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(b) $n^2 = 2\binom{n}{2} + n$

(c) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").

(d) $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$
Hint: Consider a generalization of the previous part.

3 Probability Potpourri

Prove a brief justification for each part.

(a) For two events A and B in any probability space, show that $\Pr(A \setminus B) \geq \Pr(A) - \Pr(B)$.

(b) If $|\Omega| = n$, how many distinct events does the probability space have?

(c) Find some probability space Ω and three events A, B, and $C \subseteq \Omega$ such that $\Pr(A) > \Pr(B)$ and $\Pr(A \mid C) < \Pr(B \mid C)$.

(d) If two events C and D are disjoint and $\Pr(C) > 0$ and $\Pr(D) > 0$, can C and D be independent? If so, provide an example. If not, why not?

(e) Suppose $\Pr(D \mid C) = \Pr(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

4 Parking Lots

Some of the CS 70 staff members founded a start-up company, and you just got hired. The company has twelve employees (including yourself), each of whom drive a car to work, and twelve parking spaces arranged in a row. You may assume that each day all orderings of the twelve cars are equally likely.

(a) On any given day, what is the probability that you park next to Professor Rao, who is working there for the summer?

(b) What is the probability that there are exactly three cars between yours and Professor Rao's?

(c) Suppose that, on some given day, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly five of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

5 Calculate These... or Else

(a) A straight is defined as a 5 card hand such that the card values can be arranged in consecutive ascending order, i.e. $\{8,9,10,J,Q\}$ is a straight. Values do not loop around, so $\{Q,K,A,2,3\}$ is not a straight. When drawing a 5 card hand, what is the probability of drawing a straight from a standard 52-card deck?

(b) When drawing a 5 card hand, what is the probability of drawing at least one card from each suit?

(c) Two squares are chosen at random on 8×8 chessboard. What is the probability that they share a side?

(d) 8 rooks are placed randomly on an 8×8 chessboard. What is the probability none of them are attacking each other? (Two rooks attack each other if they are in the same row, or in the same column).

(e) A bag has two quarters and a penny. If someone removes a coin, the Coin-Replenisher will come and drop in 1 of the coin that was just removed with $3/4$ probability and with $1/4$ probability drop in 1 of the opposite coin. Someone removes one of the coins at random. The Coin-Replenisher drops in a penny. You randomly take a coin from the bag. What is the probability you take a quarter?

6 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ are necessarily independent.

(b) Prove or disprove: A and $\overline{B}$ are necessarily independent.

(c) Prove or disprove: A and $\overline{A}$ are necessarily independent.

(d) Prove or disprove: It is possible that $A = B$.

7 Bag of Coins

Your friend Forest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forest picks a fair coin, and let B be the event that Forest picks a biased coin. Forest draws three coins from the bag, but he does not know which are biased and which are fair.

(a) What is the probability of FFB?

(b) What is the probability that the third coin he draws is biased?

(c) What is the probability of picking at least two fair coins?

(d) Given that Forest flips the second coin and sees heads, what is the probability that this coin is biased?