CS 70 Discrete Mathematics and Probability Theory
Spring 2020

HW 1

1 Homework Process and Study Group

Responding to this set of questions is required.

2 Propositional Logic Language

For each of the following sentences, use the notation introduced in class to convert the sentence into propositional logic. Then write the statement's negation in propositional logic without the use of the $\neg$ symbol.

(a) The cube of a negative integer is negative.

(b) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(c) There is one and only one real solution to the equation $x^3 + x + 1 = 0$.

(d) For any two distinct real numbers, we can find a rational number in between them.

Solution:

(a) We can rephrase the sentence as "if $n < 0$, then $n^3 < 0$", which can be written as

$$(\forall n \in \mathbb{Z})((n < 0) \implies (n^3 < 0))$$

or equivalently as

$$(\forall n \in \mathbb{Z})((n \geq 0) \vee (n^3 < 0)).$$

The latter is easier to negate, and its negation is given by

$$(\exists n \in \mathbb{Z})((n < 0) \wedge (n^3 \geq 0))$$

(b) The sentence is

$$(\forall x, y \in \mathbb{Z})(x^2 - y^2 \neq 10).$$

The negation is

$$(\exists x, y \in \mathbb{Z})(x^2 - y^2 = 10)$$

(c) Let $p(x) = x^3 + x + 1$. The sentence can be read "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x". Or,

$$(\exists x \in \mathbb{R})\,((p(x) = 0) \wedge ((\forall y \in \mathbb{R})(p(y) = 0) \implies (x = y)))\,.$$

Its negation is given by

$$(\forall x \in \mathbb{R})\,((p(x) \neq 0) \vee ((\exists y \in \mathbb{R})(p(y) = 0) \wedge (x \neq y)))\,.$$

This can be equivalently expressed as

$$(\forall x \in \mathbb{R})((p(x) = 0) \implies ((\exists y \in \mathbb{R})(p(y) = 0) \wedge (x \neq y))).$$

(d) The sentence can be read "if x and y are distinct real numbers, then there is a rational number z between x and y." Or,

$$(\forall x, y \in \mathbb{R})((x \neq y) \implies ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x))).$$

Equivalently,

$$(\forall x, y \in \mathbb{R})(x = y) \vee ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \wedge (z < y)$, so the above statement is equivalent to

$$(\forall x, y \in \mathbb{R})(x = y) \vee ((\exists z \in \mathbb{Q})\,((x < z) \wedge (z < y)) \vee ((y < z) \wedge (z < x))).$$

Then the negation is

$$(\exists x, y \in \mathbb{R})(x \neq y) \wedge ((\forall z \in \mathbb{Q})((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

3 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \;\equiv\; P \wedge Q$

(b) $(P \Rightarrow Q) \Rightarrow R \;\equiv\; P \Rightarrow (Q \Rightarrow R)$

(c) $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) \;\equiv\; P \Rightarrow (Q \Rightarrow R)$

(d) $(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \;\equiv\; (Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$

Solution:

(a) Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

(b) Not equivalent.

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

(c) Equivalent.

P	Q	R	$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

(d) Equivalent.

P	Q	$(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$	$(Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	F	F

4 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$.

(c) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(d) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(c) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(d) True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.

5 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then $n^2 + 1$ is even.

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x + y - |x - y|)/2$.

(c) Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(d) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Solution:

(a) We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 1$, we get $(2k+1)^2 + 1$. Simplifying the expression yields $4k^2 + 4k + 2$. This can be rewritten as $2 \times (2k^2 + 2k + 1)$. Since $2k^2 + 2k + 1$ is a natural number, by the definition of even numbers, $n^2 + 1$ is even.

(b) We will use a proof by cases. We know the following about the absolute value function for real number z.

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x - y| = y - x$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - y + x}{2} = x = \min(x,y).$$

Case 2: $x \geq y$. This means $|x - y| = x - y$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - x + y}{2} = y = \min(x,y).$$

(c)

$$\sum_{i=1}^{n} i = 1+2+\cdots+n$$
$$2\sum_{i=1}^{n} i = (1+n)+(2+(n-1))+\cdots+(n+1) = (n+1)n$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We can also do a proof by induction.
Base case: for $n=1$, $\sum i = 1^1 k = 1$.
Inductive hypothesis: Suppose $\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$.
Inductive step: We want to show $\sum_{i=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$. We have

$$\begin{aligned}\sum_{i=1}^{n+1} k &= n+1+\sum_{i=1}^{n} k \\ &= n+1+\frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}\end{aligned}$$

(d) Suppose $A' \in \mathscr{P}(A)$, that is, $A' \subseteq A$ (by the definition of the power set). We must prove that for any such A', we also have that $A' \in \mathscr{P}(B)$, that is, $A' \subseteq B$.

Let $x \in A'$. Then, since $A' \subseteq A$, $x \in A$. Since $A \subseteq B$, $x \in B$. We have shown $(\forall x \in A')\ x \in B$, so $A' \subseteq B$.

Since the previous argument works for any $A' \subseteq A$, we have proven $(\forall A' \in \mathscr{P}(A))\ A' \in \mathscr{P}(B)$. So, we conclude $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ as desired.

6 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

Solution:

(a) First we note that any integer can be written in one of the forms $3k$, $3k+1$, or $3k+2$. (Note that $3k+2$ is equal to $3(k+1)-1$. Since k is arbitary, we can treat these as equivalent forms).

We can now prove the contrapositive: that any integer $m > 3$ of the form $3k$ must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

(b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes > 5?

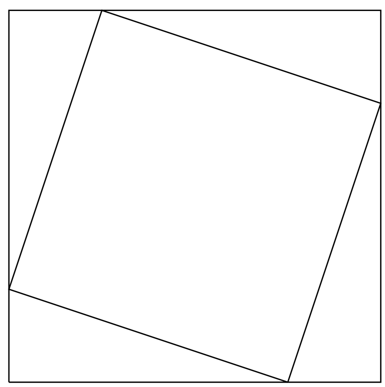
For any prime $m > 5$, we can check if $m+2$ and $m-2$ are both prime. Note that if $m > 5$, then $m+2 > 3$ and $m-2 > 3$ so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

Case 1: m is of the form $3k+1$. Then $m+2 = 3k+3$, which is divisible by 3. So $m+2$ is not prime.

Case 2: m is of the form $3k-1$. Then $m-2 = 3k-3$, which is divisible by 3. So $m-2$ is not prime.

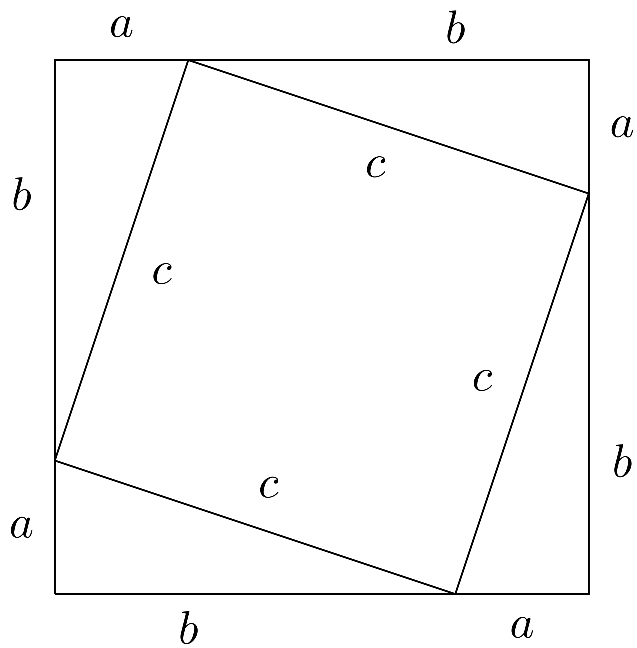
So in either case, at least one of $m+2$ and $m-2$ is not prime.

7 Pythagorean Theorem



Using the above diagram, prove the Pythagorean Theorem: if a and b are the lengths of the legs of a right triangle and c is the length of its hypotenuse, then $a^2 + b^2 = c^2$.
Hint: Look for right triangles in the diagram and label them with a, b, and c.

Solution:



Note that the large square consists of four equally sized right triangles and a small square. The area of the large square is $(a+b)^2$, the area of each right triangle is $ab/2$ and the area of the small square is c^2, so we have

$$\begin{aligned}(a+b)^2 &= 4\frac{ab}{2}+c^2 \\ a^2+2ab+b^2 &= 2ab+c^2 \\ a^2+b^2 &= c^2\end{aligned}$$

8 Inductive Charging Lemma

There are n cars on a circular track. Among all of them, they have exactly enough fuel (in total) for one car to circle the track.
Prove, using whatever method you want, that there exists at least one car that has enough fuel to reach the next car along the track.

Solution: By contradiction: If not, then they wouldn't have enough fuel in total for one car to complete the track.

Formally: Order cars clockwise around the track, starting at some arbitrary car. Let the fuel in car i be f_i liters, where 1 liter of gas corresponds to 1 kilometer of travel. Let the track be D kilometers around, and let the distance between car i and car $i+1$ (in the modular sense) be d_i kilometers. Assume, for the sake of contradiction, that $f_i < d_i$ for all i (that is, no car can reach the next car). Then $\sum_{i=1}^{n} f_i < \sum_{i=1}^{n} d_i = D$. Contradiction (the cars must have enough fuel in total for one car to circle the track).

9 Triangle Inequality

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Assuming the above inequality holds, use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Solution:

We use induction on $n \geq 2$. The base case $n = 2$ is the usual triangle inequality. Assume the inequality holds for some $n \geq 2$ (this is the inductive hypothesis). Mathematically, that is:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

For $n+1$, we want to prove:

$$|x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

Applying the original triangle inequality to the real numbers $x_1 + x_2 + \cdots + x_n$ and x_{n+1}, we get:

$$|(x_1 + x_2 + \cdots + x_n) + x_{n+1}| \leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}|$$

Adding $|x_{n+1}|$ to both sides of the inequality from the **induction hypothesis**, we get:

$$|x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

Combining these two inequalities, we have:

$$\therefore |x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

This completes the induction.

10 Counterfeit Coins

(a) Suppose you have 9 gold coins that look identical, but you also know one (and only one) of them is counterfeit. The counterfeit coin weighs slightly less than the others. You also have access to a balance scale to compare the weight of two sets of coins — i.e., it can tell you whether one set of coins is heavier, lighter, or equal in weight to another (and no other information). However, your access to this scale is very limited.

Can you find the counterfeit coin using *just two weighings*? Prove your answer.

(b) Now consider a generalization of the same scenario described above. You now have 3^n coins, $n \geq 1$, only one of which is counterfeit. You wish to find the counterfeit coin with just n weighings. Can you do it? Prove your answer.

Solution:

(a) Yes. We provide a constructive proof.

Divide this set of coins into 3 subsets of 3 each. Select two of these subsets to weigh on the balance scale. If one subset is lighter than the other, that must be the one with the counterfeit coin. If both are equal weight, the third subset must contain the counterfeit coin.

Now from this subset of 3 coins, select two coins, put one each on either side of the balance scale. If one side is lighter, that's the counterfeit coin. If both equal, the third coin is counterfeit.

(b) Proof by induction.

Base case. Select two coins, put one each on either side of the balance scale. If one side is lighter, that's the counterfeit coin. If both equal, the third coin is counterfeit.

Induction step. Assume for 3^n coins, the counterfeit coin can be detected in n weighings. Now consider 3^{n+1} coins. Divide this set of coins into 3 subsets of 3^n each. Select two of these subsets to weigh on the balance scale. If one subset is lighter than the other, that must be the one with the counterfeit coin. If both are equal weight, the third subset must contain the counterfeit coin.

From the induction hypothesis, you can now detect the counterfeit coin from the identified subset in n weighings. Thus we have $n+1$ weighings overall.