CS 70 Discrete Mathematics and Probability Theory
Fall 2019 Alistair Sinclair and Yun S. Song **HW 1**

Due: Tuesday, September 10, 2019 at 10:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

Note: *This homework consists of two parts. The first part (questions 1-5) will be graded and will determine your score for this homework. The second part (questions 6-8) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.*

Part 1: Required Problems

1 Propositional Practice

In parts (a)-(c), convert the English sentences into propositional logic. In parts (d)-(f), convert the propositions into English. In part (f), let $P(a)$ represent the proposition that a is prime.

(a) There is one and only one real solution to the equation $x^2 = 0$.

(b) Between any two distinct rational numbers, there is another rational number.

(c) If the square of an integer is greater than 4, that integer is greater than 2 or it is less than -2.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x, y \in \mathbb{Z})(x^2 - y^2 \neq 10)$

(f) $(\forall x \in \mathbb{N})\ [\ (x > 1) \implies (\exists a, b \in \mathbb{N})\ ((a + b = 2x) \wedge P(a) \wedge P(b))\ ]$

2 Tautologies and Contradictions

Classify each statement as being one of the following, where P and Q are arbitrary propositions:

- True for all combinations of P and Q (Tautology)
- False for all combinations of P and Q (Contradiction)
- Neither

Justify your answers with a truth table.

(a) $P \implies (Q \land P) \lor (\neg Q \land P)$

(b) $(P \lor Q) \lor (P \lor \neg Q)$

(c) $P \land (P \implies \neg Q) \land (Q)$

(d) $(\neg P \implies Q) \implies (\neg Q \implies P)$

(e) $(\neg P \implies \neg Q) \land (P \implies \neg Q) \land (Q)$

(f) $(\neg(P \land Q)) \land (P \lor Q)$

3 Implication

Which of the following assertions are true no matter what proposition Q represents? For any false assertion, state a counterexample (i.e. come up with a statement $Q(x,y)$ that would make the implication false). For any true assertion, give a brief explanation for why it is true.

(a) $\exists x \exists y Q(x,y) \implies \exists y \exists x Q(x,y)$.

(b) $\forall x \exists y Q(x,y) \implies \exists y \forall x Q(x,y)$.

(c) $\exists x \forall y Q(x,y) \implies \forall y \exists x Q(x,y)$.

(d) $\exists x \exists y Q(x,y) \implies \forall y \exists x Q(x,y)$.

4 Prove or Disprove

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

5 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

Note: *This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.*

Part 2: Optional Problems

6 Social Network

Suppose that $p_1, p_2, \ldots, p_n$ denote n people where every two people are either friends or strangers. Let $\text{Friends}(x,y)$ be the predicate "x and y are friends". Prove or provide a counterexample for the following statements.

(a) For all cases with $n = 5$ people, there exists a group of 3 people that are either all friends or all strangers. In mathematical notation we write this as:

$$\exists (i,j,k) \in \{1,2,\ldots,5\}^3 \text{ such that } i<j<k \text{ and } (\text{Friends}(p_i,p_j) \wedge \text{Friends}(p_j,p_k) \\ \wedge \text{Friends}(p_i,p_k)) \vee (\neg\text{Friends}(p_i,p_j) \wedge \neg\text{Friends}(p_j,p_k) \wedge \neg\text{Friends}(p_i,p_k)).$$

(b) For all cases with $n = 6$ people, there exists a group of 3 people that are either all friends or all strangers. In mathematical notation we write this as:

$$\exists (i,j,k) \in \{1,2,\ldots,6\}^3 \text{ such that } i<j<k \text{ and } (\text{Friends}(p_i,p_j) \wedge \text{Friends}(p_j,p_k) \\ \wedge \text{Friends}(p_i,p_k)) \vee (\neg\text{Friends}(p_i,p_j) \wedge \neg\text{Friends}(p_j,p_k) \wedge \neg\text{Friends}(p_i,p_k)).$$

7 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(d) $f(A \cup B) = f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(f) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

8 A Weighty Proof

You have 10 bags, each containing 100 coins. Nine of the 10 bags contain genuine gold coins, whereas one bag contains fake coins that are visually indistinguishable from the real gold coins. You don't know which bag has the fake coins, but you do know that real gold coins weigh 10g each while fake ones weigh 10.001g each. You can open the bags, look inside them, take out a few coins, mix them up, etc. You have a weighing machine that you can use *exactly once* – on which you can place a bunch of coins, press a button, and obtain a printed slip showing the weight of the coins placed, down to the milligram.

Prove that this setup is sufficient to determine which bag has the fake coins.