For each problem, justify all your answers unless otherwise specified.

Part 1: Required Problems

1 Random Variables Warm-Up

Let $X$ and $Y$ be random variables, each taking values in the set $\{0, 1, 2\}$, with joint distribution

\[
\begin{align*}
\mathbb{P}[X = 0, Y = 0] &= 1/3 \\
\mathbb{P}[X = 1, Y = 0] &= 0 \\
\mathbb{P}[X = 2, Y = 0] &= 1/9 \\
\mathbb{P}[X = 0, Y = 1] &= 0 \\
\mathbb{P}[X = 1, Y = 1] &= 1/9 \\
\mathbb{P}[X = 2, Y = 1] &= 1/9 \\
\mathbb{P}[X = 0, Y = 2] &= 1/3 \\
\mathbb{P}[X = 1, Y = 2] &= 0 \\
\mathbb{P}[X = 2, Y = 2] &= 0.
\end{align*}
\]

(a) What are the marginal distributions of $X$ and $Y$?

(b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

(c) What are $\text{var}(X)$ and $\text{var}(Y)$?

(d) Let $I$ be the indicator that $X = 1$, and $J$ be the indicator that $Y = 1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

(e) In general, let $I_A$ and $I_B$ be the indicators for events $A$ and $B$ in a probability space $(\Omega, \mathbb{P})$. What is $\mathbb{E}[I_AI_B]$, in terms of the probability of some event?

Solution:

(a) By the law of total probability

\[
\mathbb{P}[X = 0] = \mathbb{P}[X = 0, Y = 0] + \mathbb{P}[X = 0, Y = 1] + \mathbb{P}[X = 0, Y = 2] = 1/3 + 0 + 1/3 = 2/3
\]
and similarly
\[ P[X = 1] = 0 + 1/9 + 0 = 1/9 \]
\[ P[X = 2] = 1/9 + 1/9 + 0 = 2/9. \]

As a sanity check, these three numbers are all positive and they add up to \( 2/3 + 1/9 + 2/9 = 1 \) as they should. The same kind of calculation gives
\[ P[X = 2] = 1/9 + 1/9 + 0 = 2/9. \]

(b) From the above marginal distributions, we can compute
\[ E[X] = 0P[X = 0] + 1P[X = 1] + 2P[X = 2] = 5/9 \]
\[ E[Y] = 0P[Y = 0] + 1P[Y = 1] + 2P[Y = 2] = 8/9 \]

(c) Again using our marginal distributions,
\[ E[X^2] = 0P[X = 0] + 1P[X = 1] + 4P[X = 2] = 1 \]
\[ E[Y^2] = 0P[Y = 0] + 1P[Y = 1] + 4P[Y = 2] = 14/9 \]

and thus
\[ \text{var}(X) = E[X^2] - E[X]^2 = 66/81 \]
\[ \text{var}(Y) = E[Y^2] - E[Y]^2 = 62/81. \]

(d) We know that taking the expectation of an indicator for some event gives the expectation of that event, so
\[ E[I] = P[X = 1] = 1/9 \]
\[ E[J] = P[Y = 1] = 2/9. \]

The random variable \( IJ \) is equal to one if \( I = 1 \) and \( J = 1 \), and is zero otherwise. In other words, it is the indicator for the event that \( I = 1 \) \textit{and} \( J = 1 \):
\[ E[IJ] = P[I = 1, J = 1] = 1/9. \]

(e) By what we said in the previous part of the solution, \( I_A I_B \) is the indicator for the event \( A \cap B \), so
\[ E[I_A I_B] = P[A \cap B]. \]
2 Marginals

(a) Can there exist three random variables $X_1, X_2, X_3$, each taking values in the set $\{+1, -1\}$, with the property that for every $i \neq j$, the joint distribution of $X_i$ and $X_j$ is given by

$$P[X_i = 1, X_j = -1] = \frac{1}{2} \quad P[X_i = -1, X_j = 1] = \frac{1}{2} \quad P[X_i = X_j] = 0? \tag{1}$$

If so, specify the joint distribution of $X_1, X_2, X_3$; if not, prove it.

(b) For which natural numbers $n \geq 3$ can there exist random variables $X_1, X_2, \ldots, X_n$, each taking values in the set $\{+1, -1\}$, with the property that for every $i$ and $j$ satisfying $i - j = 1 \pmod{n}$, the joint distribution of $X_i$ and $X_j$ is given by (1)? For any $n$ that work, specify the joint distribution; for those that do not, prove it.

**Solution:**

(a) No such random variables can exist; let’s prove it by contradiction. From the desired joint distribution of $X_1$ and $X_2$, we claim that $X_1 = -X_2$ (by which we mean that for every $\omega$ in the sample space $X_1(\omega) = -X_2(\omega)$). Similarly, we would need to have $X_2 = -X_3$ and $X_3 = -X_1$. But now

$$X_1 = -X_2 = X_3 = -X_1,$$

a contradiction since $X_1 \in \{+1, -1\}$.

(b) This is only possible if $n$ is even. When $n = 2k + 1$, the same argument as above gives us

$$X_1 = -X_2 = X_3 = \cdots = -X_{2k} = X_{2k+1} = -X_1,$$

a contradiction for the same reason as before. However, when $n = 2k$, we can set $X_1, \ldots, X_{2k}$ to have the joint distribution

$$P[X_1 = 1, X_2 = -1, \ldots, X_{2k} = -1] = 1/2 \quad P[X_1 = -1, X_2 = 1, \ldots, X_{2k} = 1] = 1/2.$$

3 Random Tournaments

A tournament is a directed graph in which every pair of vertices has exactly one directed edge between them—for example, here are two tournaments on the vertices $\{1, 2, 3\}$:

![Tournaments](image-url)
In the first tournament above, \((1, 2, 3)\) is a Hamiltonian path, since it visits all the vertices exactly once, without repeating any edges, but \((1, 2, 3, 1)\) is not a valid Hamiltonian cycle, because the tournament contains the directed edge \(1 \rightarrow 3\) and not \(3 \rightarrow 1\). In the second tournament, \((1, 2, 3, 1)\) is a Hamiltonian cycle, as are \((2, 3, 1, 2)\) and \((3, 1, 2, 3)\); for this problem we’ll say that these are all different Hamiltonian cycles, since their start/end points are different.

Consider the following way of choosing a random tournament \(T\) on \(n\) vertices: independently for each (unordered) pair of vertices \(\{i, j\} \subset \{1, \ldots, n\}\), flip a coin and include the edge \(i \rightarrow j\) in the graph if the outcome is heads, and the edge \(j \rightarrow i\) if tails. What is the expected number of Hamiltonian paths in \(T\)? What is the expected number of Hamiltonian cycles?

**Solution:**

Each possible Hamiltonian path in the graph corresponds to a permutation \(\sigma\) of the numbers 1, \ldots, \(n\), where \(\sigma(1)\) is the starting vertex, \(\sigma(2)\) is the second vertex visited, etc. If we write \(I_\sigma\) for the indicator random variable that \(\sigma\) corresponds to an actual Hamiltonian cycle in \(T\), then

\[
\mathbb{E}[\# \text{ Hamiltonian Paths}] = \mathbb{E}\left[\sum_\sigma I_\sigma\right] = \sum_\sigma \mathbb{P}[\sigma\text{ is a Hamiltonian path in } T]
\]

In order for each \(\sigma\) to correspond to an actual Hamiltonian path in \(T\), the edges \(\sigma(i) \rightarrow \sigma(i+1)\), for \(i = 1, \ldots, n-1\) must all be included in the graph. Since the orientations of the edges in \(T\) are independent, with \(\sigma(i) \rightarrow \sigma(i+1)\) occurring with probability \(1/2\), the probability that they are all included is \(2^{-(n-1)}\). There are \(n!\) possible permutations, so we have

\[
\mathbb{E}[\# \text{ Hamiltonian Paths}] = \frac{n!}{2^{n-1}}.
\]

The situation for Hamiltonian cycles is similar. Each possible Hamiltonian cycle each possible cycle corresponds to a permutation \(\sigma\), but this time in order for \(\sigma\) to be a valid Hamiltonian cycle, \(T\) must include the edges \(\sigma(i) \rightarrow \sigma(i+1)\) for all \(i = 1, \ldots, n-1\), as well as the edge \(\sigma(n) \rightarrow \sigma(1)\). As above, these \(n\) edges are oriented independently of one another, so

\[
\mathbb{E}[\# \text{ Hamiltonian Cycles}] = \frac{n!}{2^n}.
\]

### 4 Triangles in Random Graphs

Let’s say we make a simple and undirected graph \(G\) on \(n\) vertices by randomly adding \(m\) edges, without replacement. In other words, we choose the first edge uniformly from all \(\binom{n}{2}\) possible edges, then the second one uniformly from among the remaining \(\binom{n}{2} - 1\) edges, etc. What is the expected number of triangles in \(G\)? (A triangle is a triplet of distinct vertices with all three edges present between them.)

**Solution:**
Let’s label our vertices 1, ... , n, and first check the probability that vertices 1, 2, 3 form a triangle. This event is described by a hypergeometric distribution with parameters (n \choose 2), 3, m: when we make the graph we are drawing the m edges from a bucket of (n \choose 2) possible edges, 3 of which are the ones connecting vertices 1, 2, and 3. Thus the probability that all three of these edges exist is

\[ P[1, 2, \text{ and } 3 \text{ form a triangle}] = \frac{3 \times (n \choose 2) - 3}{m} \]

In fact, there was nothing special about vertices 1, 2, 3 in this calculation. The probability that the three edges connecting some triplet of distinct vertices i, j, k is equal to the quantity above. Now, for each subset \{i, j, k\} ⊂ \{1, ..., n\}, let \(I_{i,j,k}\) be the indicator that these three vertices form a triangle. We then have

\[ \mathbb{E} [\text{# of triangles}] = \mathbb{E} \sum_{\{i,j,k\} \subset \{1, ..., n\}} I_{i,j,k} \]
\[ = \sum_{\{i,j,k\} \subset \{1, ..., n\}} \mathbb{E}[I_{i,j,k}] \quad \text{linearity of expectation} \]
\[ = \sum_{\{i,j,k\} \subset \{1, ..., n\}} P[i, j, \text{ and } k \text{ form a triangle}] \]
\[ = \sum_{\{i,j,k\} \subset \{1, ..., n\}} \frac{3\times (n \choose 2) - 3}{m} \]
\[ = \binom{n}{3} \frac{3\times (n \choose 2) - 3}{m} \]

5 Variance

A building has n upper floors numbered 1, 2, ... , n, plus a ground floor G. At the ground floor, m people get on the elevator together, and each person gets off at one of the n upper floors uniformly at random and independently of everyone else. What is the variance of the number of floors the elevator does not stop at?

Solution: Let \(N\) be the number of floors the elevator does not stop at. We can represent \(N\) as the sum of the indicator variables \(I_1, \ldots, I_n\), where \(I_i = 1\) if no one gets off on floor i. Thus, we have

\[ \mathbb{E}[I_i] = P[I_i = 1] = \left(\frac{n-1}{n}\right)^m, \]

and from linearity of expectation,

\[ \mathbb{E}[N] = \sum_{i=1}^{n} \mathbb{E}[I_i] = n \left(\frac{n-1}{n}\right)^m. \]
To find the variance, we cannot simply sum the variance of our indicator variables. However, since
\[ \text{var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 \]
the only piece we don’t already know is \( \mathbb{E}[N^2] \). We can calculate this by again expanding \( N \) as a sum:
\[
\mathbb{E}[N^2] = \mathbb{E}[(I_1 + \cdots + I_n)^2] = \mathbb{E}\left[ \sum_{i,j} I_i I_j \right] = \sum_{i,j} \mathbb{E}[X_i X_j] = \sum_i \mathbb{E}[I_i^2] + \sum_{i \neq j} \mathbb{E}[I_i I_j].
\]
The first term is simple to calculate: since \( I_i \) is an indicator, \( I_i^2 = I_i \), so we have
\[
\mathbb{E}[I_i^2] = \mathbb{E}[I_i] = \mathbb{P}[I_i = 1] = \left( \frac{n-1}{n} \right)^m,
\]
meaning that
\[
\sum_{i=1}^n \mathbb{E}[I_i^2] = n \left( \frac{n-1}{n} \right)^m.
\]
From the definition of the variables \( I_i \), we see that \( I_i I_j = 1 \) when both \( I_i \) and \( I_j \) are 1, which means no one gets off the elevator on floor \( i \) and floor \( j \). This happens with probability
\[
\mathbb{P}[I_i = I_j = 1] = \mathbb{P}[I_i = 1 \cap I_j = 1] = \left( \frac{n-2}{n} \right)^m.
\]
Thus we now know
\[
\sum_{i \neq j} \mathbb{E}[I_i I_j] = n(n-1) \left( \frac{n-2}{n} \right)^m,
\]
and we can assemble everything we’ve done so far to see that
\[
\text{var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = n \left( \frac{n-1}{n} \right)^m + n(n-1) \left( \frac{n-2}{n} \right)^m - n^2 \left( \frac{n-1}{n} \right)^{2m}.
\]

**Note:** This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.

---

**Part 2: Optional Problems**

6 **Indicators, Probabilities, and Positivity**

(a) Let \( X \) be a positive random variable, i.e. \( X(\omega) \geq 0 \) for every \( \omega \in \Omega \). Prove that \( \mathbb{E}[X] \geq 0 \).

(b) Let \( n \) be a natural number, \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), and let \((\Omega, \mathbb{P})\) be a probability space with some events \( A_1, \ldots, A_n \subset \Omega \). Prove that \( \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathbb{P}(A_i \cap A_j) \geq 0 \). Note that \( \alpha_i \) can be less than 0.
(c) Again let $X$ be a positive random variable, and let $I$ be the indicator that $X > 0$. Prove that

$$P[X > 0] \geq \frac{(E[X])^2}{E[X^2]}.$$ 

It may be useful to prove that $X = XI$, and to consider the random variable $(X + aI)^2$ for various values of $a \in \mathbb{R}$.

**Solution:**

(a) Directly from the definition, $E[X] = \sum_{\omega \in \Omega} P[\omega]X(\omega)$, and the right hand side is a sum of nonnegative terms.

(b) Let’s write the sum using the indicators $I_1, \ldots, I_n$ for the events $A_1, \ldots, A_n$. Then,

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j P[A_i \cap A_j] = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j E[I_i I_j] = E \left[ \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j I_i I_j \right]$$

$$= E \left[ \left( \sum_{i=1}^n \alpha_i I_i \right) \left( \sum_{i=1}^n \alpha_i I_i \right) \right] = E \left[ \left( \sum_{i=1}^n \alpha_i I_i \right)^2 \right] \geq 0.$$

(c) As suggested, note that $X = XI$, since if $X(\omega) = 0$, then $I(\omega) = 0$, and if $X(\omega) > 0$, $I(\omega) = 1$. We know that for any $a$, $(aX + I)^2$ is a positive random variable, so from the first part of the problem,

$$0 \leq E[(aX + I)^2] = a^2 E[X^2] + 2a E[XI] + E[I^2] = a^2 E[X^2] + 2a E[X] + P[X > 0].$$

Rearranging gives

$$P[X > 0] \geq -2a E[X] - a^2 E[X^2],$$

and setting $a = -E[X]/E[X^2]$ will give us the requested lower bound.

7  Swaps and Cycles

We’ll say that a permutation $\pi = (\pi(1), \ldots, \pi(n))$ contains a swap if there exist $i, j \in \{1, \ldots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$.

(a) What is the expected number of swaps in a random permutation?

(b) What about the variance?

(c) We say that $\pi$ is an involution if $\pi(\pi(i)) = i$ for every $i = 1, \ldots, n$. What is the probability that $\pi$ is an involution? The answer may depend on $n$...

(d) In the same spirit as above, we’ll say that $\pi$ contains a $s$-cycle if there exist $i_1, \ldots, i_s \in \{1, \ldots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1$. Compute the expectation and variance of the number of $s$-cycles.
Solution:

(a) As a warm-up, let’s compute the probability that 1 and 2 are swapped. There are $n!$ possible permutations, and $(n-2)!$ of them have $\pi(1) = 2$ and $\pi(2) = 1$. This means
\[
\mathbb{P}[(1, 2) \text{ are a swap}] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.
\]
There was nothing special about 1 and 2 in this calculation, so for any $\{i, j\} \subset \{1, \ldots, n\}$, the probability that $i$ and $j$ are swapped is the same as above. Let’s write $I_{i,j}$ for the indicator that $i$ and $j$ are swapped, and $N$ for the total number of swaps, so that
\[
\mathbb{E}[N] = \mathbb{E} \left[ \sum_{\{i, j\} \subset \{1, \ldots, n\}} I_{i,j} \right] = \sum_{\{i, j\} \subset \{1, \ldots, n\}} \mathbb{P}(i, j \text{ are swapped}) = \frac{1}{n(n-2)} \binom{n}{2} = \frac{1}{2}.
\]

(b) For the variance, when we expand $N^2$ as a sum over pairs $\{i, j\}, \{k, l\}$, we’ll need to know $\mathbb{E}[I_{i,j}I_{k,l}]$, which is just the probability that $i$ and $j$ are swapped and $k$ and $l$ are swapped as well. There are three cases to consider. If $\{i, j\} = \{k, l\}$, then the probability is exactly what we computed above. If the two pairs share one element, then it is impossible that they are both swaps. If the pairs are disjoint, then of the $n!$ possible permutations, $(n-4)!$ include the two swaps we are concerned with. Thus
\[
\mathbb{E}[N^2] = \sum_{\{i, j\} \subset \{1, \ldots, n\}} \mathbb{E}[I_{i,j}^2] + \sum_{\{i, j\} \cap \{k, l\} = \emptyset} \mathbb{E}[I_{i,j}I_{k,l}]
\]
\[
= \binom{n}{2} \left( \frac{1}{n(n-1)} + \binom{n-2}{2} \frac{1}{n(n-1)(n-2)(n-3)} \right)
\]
\[
= \frac{1}{2} + \frac{n(n-1)}{2} \frac{(n-2)(n-3)}{2} \frac{1}{n(n-1)(n-2)(n-3)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.
\]
And $\text{var}(N^2) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$.

(c) Let $\pi$ be an involution, and consider some $i \in \{1, \ldots, n\}$. If we call $j = \pi(i)$, we know from the definition that $i = \pi(\pi(i)) = \pi(j)$. Thus every $i = 1, \ldots, n$ belongs to a swap, and consequently we can partition the set $\{1, \ldots, n\}$ into swaps (this tells us right away that $n$ is even). To count involutions, then, we can alternatively count matchings of the set $\{1, \ldots, n\}$, i.e. partitions of this set into $n/2$ subsets of size 2. There are
\[
\binom{n}{2} \binom{n-2}{2} \cdots \binom{4}{2} \binom{2}{2} = \frac{n!}{2^{n/2}}
\]
such matchings, so the probability that a random permutation is an involution is $2^{-n/2}$.

(d) The idea here is quite similar to the above, so we’ll be a little less verbose in the exposition. However, as a first aside we need the notion of a cyclic ordering of $s$ elements from a set $\{1, \ldots n\}$. We mean by this a labelling of the $s$ beads of a necklace with elements of the set, where we say that labelings of the beads are the same if we can move them along the string to
turn one into the other. For example, \((1, 2, 3, 4)\) and \((1, 2, 4, 3)\) are different cyclic orderings, but \((1, 2, 3, 4)\) and \((2, 3, 4, 1)\) are the same. There are

\[
\binom{n}{s} \frac{s!}{s} = \frac{n!}{(n-s)! \cdot s}
\]

possible cyclic orderings of length \(s\) from a set with \(n\) elements, since if we first count all subsets of size \(s\), and then all permutations of each of those subsets, we have overcounted by a factor of \(s\).

Now, let \(N\) be a random variable counting the number of \(s\)-cycles, and for each cyclic ordering \((i_1, \ldots, i_s)\) of \(s\) elements of \(\{1, \ldots, n\}\), let \(I_{(i_1, \ldots, i_s)}\) be the indicator that \(\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1\). There are \((n-s)!\) permutations in which \((i_1, \ldots, i_s)\) form an \(s\)-cycle (since we are free to do whatever we want to the remaining \((n-s)\) elements of \(\{1, \ldots, n\}\)), so the probability that \((i_1, \ldots, i_s)\) are such a cycle is \(\frac{(n-s)!}{n!}\), and

\[
\mathbb{E}[N] = \mathbb{E} \left[ \sum_{(i_1, \ldots, i_s) \text{ cyclic ordering}} I_{(i_1, \ldots, i_s)} \right] = \frac{n!}{(n-s)!} \cdot \frac{1}{s} \cdot \frac{(n-s)!}{n!} = \frac{1}{s}.
\]

For the variance, we need to know \(\mathbb{E}[I_{(i_1, \ldots, i_s)}I_{(j_1, \ldots, j_s)}]\), the probability that both \((i_1, \ldots, i_s)\) and \((j_1, \ldots, j_s)\) are \(s\)-cycles. We have already computed this probability if the two cyclic orderings are the same, and we know that it is zero if they overlap but are not equal. If they are disjoint, then there are \((n-2s)!\) permutations in which both are \(s\)-cycles, so \(\mathbb{E}[I_{(i_1, \ldots, i_s)}I_{(j_1, \ldots, j_s)}] = \frac{(n-2s)!}{n!}\), and there are

\[
\frac{n!}{(n-s)!} \cdot \frac{(n-s+1)!}{(n-2s)!} \cdot \frac{1}{s^2}
\]

ways to choose the two disjoint cyclic orderings, so mirroring the calculation earlier,

\[
\mathbb{E}[N^2] = \frac{1}{s} + \frac{1}{s^2},
\]

and \(\text{var}(N) = \frac{1}{s}\).