Part 1: Required Problems

1 Random Variables Warm-Up

Let $X$ and $Y$ be random variables, each taking values in the set $\{0, 1, 2\}$, with joint distribution

\[
\begin{align*}
\mathbb{P}[X = 0, Y = 0] &= 1/3 & \mathbb{P}[X = 0, Y = 1] &= 0 & \mathbb{P}[X = 0, Y = 2] &= 1/3 \\
\mathbb{P}[X = 1, Y = 0] &= 0 & \mathbb{P}[X = 1, Y = 1] &= 1/9 & \mathbb{P}[X = 1, Y = 2] &= 0 \\
\mathbb{P}[X = 2, Y = 0] &= 1/9 & \mathbb{P}[X = 2, Y = 1] &= 1/9 & \mathbb{P}[X = 2, Y = 2] &= 0.
\end{align*}
\]

(a) What are the marginal distributions of $X$ and $Y$?

(b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

(c) What are $\text{var}(X)$ and $\text{var}(Y)$?

(d) Let $I$ be the indicator that $X = 1$, and $J$ be the indicator that $Y = 1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

(e) In general, let $I_A$ and $I_B$ be the indicators for events $A$ and $B$ in a probability space $(\Omega, \mathbb{P})$. What is $\mathbb{E}[I_AI_B]$, in terms of the probability of some event?
2 Marginals

(a) Can there exist three random variables $X_1, X_2, X_3$, each taking values in the set $\{+1, -1\}$, with the property that for every $i \neq j$, the joint distribution of $X_i$ and $X_j$ is given by

$$\mathbb{P}[X_i = 1, X_j = -1] = \frac{1}{2} \quad \mathbb{P}[X_i = -1, X_j = 1] = \frac{1}{2} \quad \mathbb{P}[X_i = X_j] = 0? \quad (1)$$

If so, specify the joint distribution of $X_1, X_2, X_3$; if not, prove it.

(b) For which natural numbers $n \geq 3$ can there exist random variables $X_1, X_2, \ldots, X_n$, each taking values in the set $\{+1, -1\}$, with the property that for every $i$ and $j$ satisfying $i - j = 1 \pmod{n}$, the joint distribution of $X_i$ and $X_j$ is given by (1)? For any $n$ that work, specify the joint distribution; for those that do not, prove it.

3 Random Tournaments

A tournament is a directed graph in which every pair of vertices has exactly one directed edge between them—for example, here are two tournaments on the vertices $\{1, 2, 3\}$:

In the first tournament above, $(1, 2, 3)$ is a Hamiltonian path, since it visits all the vertices exactly once, without repeating any edges, but $(1, 2, 3, 1)$ is not a valid Hamiltonian cycle, because the tournament contains the directed edge $1 \to 3$ and not $3 \to 1$. In the second tournament, $(1, 2, 3, 1)$ is a Hamiltonian cycle, as are $(2, 3, 1, 2)$ and $(3, 1, 2, 3)$; for this problem we’ll say that these are all different Hamiltonian cycles, since their start/end points are different.

Consider the following way of choosing a random tournament $T$ on $n$ vertices: independently for each (unordered) pair of vertices $\{i, j\} \subset \{1, \ldots, n\}$, flip a coin and include the edge $i \to j$ in the graph if the outcome is heads, and the edge $j \to i$ if tails. What is the expected number of Hamiltonian paths in $T$? What is the expected number of Hamiltonian cycles?

4 Triangles in Random Graphs

Let’s say we make a simple and undirected graph $G$ on $n$ vertices by randomly adding $m$ edges, without replacement. In other words, we choose the first edge uniformly from all $\binom{n}{2}$ possible edges, then the second one uniformly from among the remaining $\binom{n}{2} - 1$ edges, etc. What is the expected number of triangles in $G$? (A triangle is a triplet of distinct vertices with all three edges present between them.)
5 Variance

A building has $n$ upper floors numbered $1, 2, \ldots, n$, plus a ground floor $G$. At the ground floor, $m$ people get on the elevator together, and each person gets off at one of the $n$ upper floors uniformly at random and independently of everyone else. What is the variance of the number of floors the elevator does not stop at?

Note: This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.

Part 2: Optional Problems

6 Indicators, Probabilities, and Positivity

(a) Let $X$ be a positive random variable, i.e. $X(\omega) \geq 0$ for every $\omega \in \Omega$. Prove that $\mathbb{E}[X] \geq 0$.

(b) Let $n$ be a natural number, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, and let $(\Omega, \mathbb{P})$ be a probability space with some events $A_1, \ldots, A_n \subset \Omega$. Prove that $\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \mathbb{P}(A_i \cap A_j) \geq 0$. Note that $\alpha_i$ can be less than 0.

(c) Again let $X$ be a positive random variable, and let $I$ be the indicator that $X > 0$. Prove that

$$\mathbb{P}[X > 0] \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$ 

It may be useful to prove that $X = XI$, and to consider the random variable $(X + aI)^2$ for various values of $a \in \mathbb{R}$.

7 Swaps and Cycles

We’ll say that a permutation $\pi = (\pi(1), \ldots, \pi(n))$ contains a swap if there exist $i, j \in \{1, \ldots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$.

(a) What is the expected number of swaps in a random permutation?

(b) What about the variance?

(c) We say that $\pi$ is an involution if $\pi(\pi(i)) = i$ for every $i = 1, \ldots, n$. What is the probability that $\pi$ is an involution? The answer may depend on $n$...

(d) In the same spirit as above, we’ll say that $\pi$ contains a $s$-cycle if there exist $i_1, \ldots, i_s \in \{1, \ldots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1$. Compute the expectation and variance of the number of $s$-cycles.