Sundry

Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of homework party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

Please copy the following statement and sign next to it:

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.

1 More Family Planning

(a) Suppose we have a random variable $N \sim Geom(1/3)$ representing the number of children of a randomly chosen family. Assume that within the family, children are equally likely to be boys and girls. Let B be the number of boys and G the number of girls in the family. What is the joint probability distribution of B, G?

(b) Given that we know there are 0 girls in the family, what is the most likely number of boys in the family?

(c) Now let X and Y be independent random variables representing the number of children in two independently, randomly chosen families. Suppose $X \sim Geom(p)$ and $Y \sim Geom(q)$. Using their joint distribution, find the probability that the number of children in the first family (X) is less than the number of children in the second family (Y). (You may use the convergence formula for a Geometric Series: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$)

(d) Show how you could obtain your answer from the previous part using an interpretation of the geometric distribution.

2 Uniform Means

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed uniform random variables on the interval $[0,1]$ (where n is a positive integer).

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Z)$. [*Hint*: Find the CDF.]

3 Variance of the Minimum of Uniform Random Variables

Let n be a positive integer and let $X_1, \ldots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Uniform}[0,1]$. Find $\operatorname{var} Y$, where

$$Y := \min\{X_1, \ldots, X_n\}.$$

(Hint: If you get stuck with the integral for $\mathbb{E}[Y^2]$, try reviewing how to perform integration by parts.)

4 Arrows

You and your friend are competing in an archery competition. You are a more skilled archer than he is, and the distances of your arrows to the center of the bullseye are i.i.d. Uniform$[0,1]$ whereas his are i.i.d. Uniform$[0,2]$. To even out the playing field, you both agree that you will shoot one arrow and he will shoot two. The arrow closest to the center of the bullseye wins the competition. What is the probability that you will win? *Note*: The distances *from the center of the bullseye* are uniform.

5 Darts (Again!)

Alvin is playing darts. His aim follows an exponential distribution; that is, the probability density that the dart is x distance from the center is $f_X(x) = \exp(-x)$. The board's radius is 4 units.

(a) What is the probability the dart will stay within the board?

(b) Say you know Alvin made it on the board. What is the probability he is within 1 unit from the center?

(c) If Alvin is within 1 unit from the center, he scores 4 points, if he is within 2 units, he scores 3, etc. In other words, Alvin scores $\lfloor 5 - x \rfloor$, where x is the distance from the center. What is Alvin's expected score after one throw?

6 Exponential Distributions: Lightbulbs

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?