Note: This homework consists of two parts. The first part (questions 1-5) will be graded and will determine your score for this homework. The second part (questions 6-8) will be graded if you submit them, but will not affect your homework score in any way. You are strongly advised to attempt all the questions in the first part. You should attempt the problems in the second part only if you are interested and have time to spare.

For each problem, justify all your answers unless otherwise specified.

Part 1: Required Problems

1 Warm-up

Give numerical answers; no justification necessary.

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d. and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of $X$.

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3. Calculate the probability that there is at least one accident this week.

(d) Consider an experiment that consists of counting the number of $\alpha$ particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such $\alpha$-particles are given off per second, what is a good approximation to the probability that no more that 2 $\alpha$-particles will appear?

Solution:

(a) 0.0461

Let $N$ denote the random variable that you hit the center on your $X$-th turn. Then $X \sim \text{Geometric}(0.17)$ and hence,

$$\mathbb{P}(X = 8) = (0.17)(1 - 0.17)^7 \approx 0.0461.$$
(b) $\mathbb{E}(X) = 5$ and $\text{Var}(X) = 20$

This follows from $\mathbb{E}(X) = 1/p$ and $\text{Var}(X) = (1 - p)/(p^2)$ for $X \sim \text{Geometric}(p)$ as seen in Theorem 18.3 of Lecture Note 18.

(c) 0.9502

Let $X$ denote the number of accidents occurring on the stretch of highway in question during this week. We have $X \sim \text{Poisson}(3)$ and hence,

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0),$$

$$= 1 - e^{-3} \frac{3^0}{0!}$$

$$= 1 - e^{-3} \approx 0.9502.$$

(d) 0.382

We model the number of $\alpha$-particles given off during the second considered as a Poisson random variable with parameter $\lambda = 3.2$. Hence,

$$\mathbb{P}(X \leq 2) = e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2}e^{-3.2} = 0.382.$$

2 Class Enrollment

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the marine science class on each attempt is $\mu$ and the probability of enrolling successfully in CS 70 on each attempt is $\lambda$. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the marine science class everyday and gets enrolled in it on day $M$. What is the distribution of $M$?

(b) Suppose she is not enrolled in the marine science class after attempting each day for the first 5 days. What is the conditional distribution of $M$ given $M > 5$?

(c) Once she is enrolled in the marine science class, she starts attempting to enroll in CS 70 from day $M + 1$ and gets enrolled in it on day $C$. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[C]$. 

(d) Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let $M$ be the number of days it takes to enroll in the marine science class, and $C$ be the number of days it takes to enroll in CS 70. What is the distribution of $M$ and $C$ now? Are they independent?
Let $X$ denote the day she gets enrolled in her first class and let $Y$ denote the day she gets enrolled in both the classes. What is the distribution of $X$?

What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[Y]$.

What is the expected number of classes she will be enrolled in by the end of 14 days?

**Solution:**

(a) $M \sim \text{Geometric}(\mu)$.

(b) Given that $M > 5$, the random variable $M$ takes values in $\{6, 7, \ldots\}$. For $i = 6, 7, \ldots$,

$$
P[M = i | M > 5] = \frac{P[M = i \land M > 5]}{P[M > 5]} = \frac{P[M = i]}{P[M > 5]} = \frac{\mu(1 - \mu)^{i-1}}{(1 - \mu)^5} = \mu(1 - \mu)^{i-6}.
$$

If $K$ denotes the additional number of days it takes to get enrolled in the marine science class after day 5, i.e. $K = M - 5$, then conditioned on $M > 5$, the random variable $K$ has the geometric distribution with parameter $\mu$. Note that this is the same as the distribution of $M$. This is known as the memoryless property of geometric distribution.

(c) We have $C - M \sim \text{Geometric}(\lambda)$. Thus $\mathbb{E}[M] = 1/\mu$ and $\mathbb{E}[C - M] = 1/\lambda$. And hence $\mathbb{E}[C] = \mathbb{E}[M] + \mathbb{E}[C - M] = 1/\mu + 1/\lambda$.

(d) $M \sim \text{Geometric}(\mu)$, $C \sim \text{Geometric}(\lambda)$. Yes they are independent.

(e) We have $X = \min\{M, C\}$ and $Y = \max\{M, C\}$. We also use the following definition of the minimum:

$$
\min(m, c) = \begin{cases} m & \text{if } m \leq c; \\ c & \text{if } m > c. 
\end{cases}
$$

Now, for all $k \in \{1, 2, \ldots\}$, $\min(M, C) = k$ is equivalent to $(M = k) \cap (C \geq k)$ or $(C = k) \cap (M > k)$. Hence,

$$
P[X = k] = P[\min(M, C) = k] = P[(M = k) \cap (C \geq k)] + P[(C = k) \cap (M > k)]
$$

(since $M$ and $C$ are independent)

$$
= [(1 - \mu)^{k-1}\mu](1 - \lambda)^{k-1} + [(1 - \lambda)^{k-1}\lambda](1 - \mu)^k
$$

(since $M$ and $C$ are geometric)

$$
= ((1 - \mu)(1 - \lambda))^{k-1}(\mu + \lambda(1 - \mu))
$$

$$
= (1 - \mu - \lambda + \lambda\mu)^{k-1}(\mu + \lambda - \mu\lambda).
$$
But this final expression is precisely the probability that a geometric r.v. with parameter \( \mu + \lambda - \mu \lambda \) takes the value \( k \). Hence \( X \sim \text{Geom}(\mu + \lambda - \mu \lambda) \).

An alternative, slightly cleaner approach is to work with the tail probabilities of the geometric distribution, rather than with the usual point probabilities as above. In other words, we can work with \( \mathbb{P}[X \geq k] \) rather than with \( \mathbb{P}[X = k] \); clearly the values \( \mathbb{P}[X \geq k] \) specify the values \( \mathbb{P}[X = k] \) since \( \mathbb{P}[X = k] = \mathbb{P}[X \geq k] - \mathbb{P}[X \geq (k + 1)] \), so it suffices to calculate them instead. We then get the following argument:

\[
\mathbb{P}[X \geq k] = \mathbb{P}[\min(M, C) \geq k] = \mathbb{P}[(M \geq k) \cap (C \geq k)] = \mathbb{P}[M \geq k] \cdot \mathbb{P}[C \geq k] \quad \text{since } M, C \text{ are independent} \\
= (1 - \mu)^{k-1} (1 - \lambda)^{k-1} \quad \text{since } M, C \text{ are geometric} \\
= ((1 - \mu)(1 - \lambda))^{k-1} = (1 - \mu - \lambda + \mu \lambda)^{k-1}.
\]

This is the tail probability of a geometric distribution with parameter \( \mu + \lambda - \mu \lambda \), so we are done.

(f) From part (e) we get \( \mathbb{E}[X] = 1/(\mu + \lambda - \mu \lambda) \). From part (d) we have \( \mathbb{E}[M] = 1/\mu \) and \( \mathbb{E}[C] = 1/\lambda \). We now observe that \( \min\{m, c\} + \max\{m, c\} = m + c \). Using linearity of expectation we get \( \mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[M] + \mathbb{E}[C] \). Thus \( \mathbb{E}[Y] = 1/\mu + 1/\lambda - 1/(\mu + \lambda - \mu \lambda) \).

(g) Let \( I_M \) and \( I_C \) be the indicator random variables of the events "\( M \leq 14 \)" and "\( C \leq 14 \)" respectively. Then \( I_M + I_C \) is the number of classes she will be enrolled in within 14 days. Hence the answer is \( \mathbb{E}[I_M] + \mathbb{E}[I_C] = \mathbb{P}[M \leq 14] + \mathbb{P}[C \leq 14] = 1 - (1 - \mu)^{14} + 1 - (1 - \lambda)^{14} \)

3 Student Life

In an attempt to avoid having to do laundry often, Marcus comes up with a system. Every night, he designates one of his shirts as his dirtiest shirt. In the morning, he randomly picks one of his shirts to wear. If he picked the dirtiest one, he puts it in a dirty pile at the end of the day (a shirt in the dirty pile is not used again until it is cleaned). When Marcus puts his last shirt into the dirty pile, he finally does his laundry, and again designates one of his shirts as his dirtiest shirt (laundry isn’t perfect) before going to bed. This process then repeats.

(a) If Marcus has \( n \) shirts, what is the expected number of days that transpire between laundry events? Your answer should be a function of \( n \) involving no summations.

(b) Say he gets even lazier, and instead of organizing his shirts in his dresser every night, he throws his shirts randomly onto one of \( n \) different locations in his room (one shirt per location), designates one of his shirts as his dirtiest shirt, and one location as the dirtiest location. In the morning, if he happens to pick the dirtiest shirt, and the dirtiest shirt was in the dirtiest location, then he puts the shirt into the dirty pile at the end of the day and does not throw any future
shirts into that location and also does not consider it as a candidate for future dirtiest locations (it is too dirty). What is the expected number of days that transpire between laundry events now? Again, your answer should be a function of \( n \) involving no summations.

**Solution:**

(a) The number of days that it takes for him to throw a shirt into the dirty pile can be represented as a geometric RV. For the first shirt, this is the geometric RV with \( p = 1/n \). We can see this by noticing that every day the probability of getting the dirtiest shirt remains \( 1/n \).

We’ll call \( X_i \) the number of days that go until he throws the \( i \)th shirt into the dirty pile. Since on the \( i \)th shirt, there are \( n - i + 1 \) shirts left, we get that \( X_i \sim \text{Geometric}(1/(n - i + 1)) \). The number of days until he does his laundry is a sum of these variables. Therefore, we can get the following result:

\[
E[X] = E\left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} (n - i + 1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]

(b) For this part we can use a similar approach but the probability for \( X_i \) becomes \( 1/(n - i + 1)^2 \). This is because the dirtiest shirt falls into the dirtiest spot with probability \( 1/(n - i + 1) \) and we pick it after that with probability \( 1/(n - i + 1) \), so the probability of picking the dirtiest shirt from the dirtiest spot for the \( i \)th shirt is \( 1/(n - i + 1)^2 \). Using the same approach, we get the following sum:

\[
E[X] = E\left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} (n - i + 1)^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

4 Unreliable Servers

In a single cluster of a Google competitor, there are a huge number of servers \( n \), each with a uniform and independent probability of going down in a given day. On average, 4 servers go down in the cluster per day. As each cluster is responsible for a huge amount of internet traffic, it is fair to assume that \( n \) is a very large number. Recall that as \( n \to \infty \), a \( \text{Binom}(n, \lambda/n) \) distribution will tend towards a \( \text{Poisson}(\lambda) \) distribution.

(a) What is an appropriate distribution to model the number of servers that crash on any given day for a certain cluster?

(b) Compute the expected value and variance of the number of crashed servers on a given day for a certain cluster.

(c) Compute the probability that fewer than 3 servers crashed on a given day for a certain cluster.
(d) Compute the probability at least 3 servers crashed on a given day for a certain cluster.

**Solution:**

(a) Because each server goes down independently of the other servers, and with the same probability, the number of servers that crash on a given day follows a binomial distribution $\text{Binom}(n, p)$, where $n$ is the number of servers and $p$ is the probability of each individual server crashing on any given day. Since on average, 4 servers crash per day, we have $p = \frac{4}{n}$. We are given that the number of servers in the cluster is large, so $n \gg p$ and we can model the number of servers that crash as a Poisson distribution with $\lambda = 4$.

(b) Recall that the expectation and variance of a Poisson distribution with parameter $\lambda$ are both equal to $\lambda$ and in this case $\lambda = 4$.

(c) To compute the probability that fewer than 3 servers went down, we must add the probabilities that 0 servers go down, 1 server goes down, and the probability that 2 servers go down. The PMF of the Poisson distribution is

$$
P[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}.
$$

Thus

$$
P[X = 0 \text{ or } X = 1 \text{ or } X = 2] = e^{-4} + 4e^{-4} + \frac{4^2}{2}e^{-4} = e^{-4} + 4e^{-4} + 8e^{-4} = 13e^{-4}.
$$

(d) $1 - P[\text{fewer than 3 servers crashed}] = 1 - 13e^{-4}$.

5 Shuttles and Taxis at Airport

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson process. The shuttles arrive at a rate $\lambda_1 = 1/20$ (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate $\lambda_2 = 1/10$ (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

(a) What is the distribution of the following:

(i) The number of taxis that arrive between times 00:00 and 00:20?
(ii) The number of shuttles that arrive between times 00:00 and 00:20?
(iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20?

(b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?

(c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?
(d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

Solution:

(a) (i) Let $T([0, 20])$ denote the number of taxis that arrive between times 00:00 and 00:20. This interval has length 20 minutes, so the number of taxis $T([0, 20])$ arriving in this interval is distributed according to Poisson$(\lambda_2 \cdot 20) = \text{Poisson}(2)$, i.e.

$$P[T([0, 20]) = t] = \frac{2^t e^{-2}}{t!}, \text{ for } t = 0, 1, 2, \ldots.$$  

(ii) Let $S([0, 20])$ denote the number of shuttles that arrive between times 00:00 and 00:20. This interval has length 20 minutes, so the number of shuttles $S([0, 20])$ arriving in this interval is distributed according to Poisson$(\lambda_1 \cdot 20) = \text{Poisson}(1)$, i.e.

$$P[S([0, 20]) = s] = \frac{1^s e^{-1}}{s!}, \text{ for } s = 0, 1, 2, \ldots.$$  

(iii) Let $N([0, 20]) = S([0, 20]) + T([0, 20])$ denote the total number of pickup vehicles (taxis and shuttles) arriving between times 00:00 and 00:20. Since the sum of independent Poisson random variables is Poisson distributed with parameter given by the sum of the individual parameters, we have $N([0, 20]) \sim \text{Poisson}(3)$, i.e.

$$P[N([0, 20]) = n] = \frac{3^n e^{-3}}{n!}, \text{ for } n = 0, 1, 2, \ldots.$$  

(b) We have

$$P[T([0, 20]) = 3] = \frac{2^3 e^{-2}}{3!} \text{ and } P[S([0, 20]) = 1] = \frac{1^1 e^{-1}}{1!}.$$  

Since the taxis and the shuttles arrive independently, the probability that exactly 3 taxis and 1 shuttle arrive in this interval is given by the product of their individual probabilities, i.e.

$$\frac{2^3 e^{-2} 1^1 e^{-1}}{3! 1!} = \frac{4}{3} e^{-3} \approx 0.0664.$$  

(c) Let $A$ be the event that exactly 1 taxi arrives between times 00:00 and 00:20. Let $B$ be the event that exactly 1 vehicle arrives between times 00:00 and 00:20. We have

$$P[B] = \frac{3^1 e^{-3}}{1!}.$$  

Event $A \cap B$ is the event that exactly 1 taxi and 0 shuttles arrive between times 00:00 and 00:20. Hence

$$P[A \cap B] = \frac{2^1 e^{-2} 1^0 e^{-1}}{1! 0!}.$$  

Thus, we get

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = 2/3.$$
(d) The event that you need to wait for more than 10 minutes starting 00:20 is equivalent to the event that no vehicle arrives between times 00:20 and 00:30. Let \( N([20, 30]) \) denote the number of vehicles that arrive between times 00:20 and 00:30. This interval has length 10 minutes, so \( N([20, 30]) \sim \text{Poisson}(\lambda_1 + \lambda_2) \cdot 10 = \text{Poisson}(3/2) \). Since Poisson arrivals in disjoint intervals are independent, we have

\[
\mathbb{P}[N([20, 30]) = 0 \mid T([0, 20]) = 3, S([0, 20]) = 1] = \mathbb{P}[N([20, 30]) = 0] \sim \frac{1.5^0 e^{-1.5}}{0!} = e^{-1.5} \approx 0.2231.
\]

**Note:** This concludes the first part of the homework. The problems below are optional, will not affect your score, and should be attempted only if you have time to spare.

---

**Part 2: Optional Problems**

6 Alternating Technicians

A faulty machine is repeatedly run and on each run, the machine fails with probability \( p \) independent of the number of runs. Let the random variable \( X \) denote the number of runs until the first failure. Now, two technicians are hired to check on the machine every run. They decide to take turns checking on the machine every run. What is the probability that the first technician is the first one to find the machine broken?

**Solution:**

Let the required probability be denoted by \( q \) (the probability that the first technician is the first to find the machine fail). Now, we have:

\[
q = \mathbb{P}[X = 1] + \mathbb{P}[X = 3] + \mathbb{P}[X = 5] + \ldots,
\]

because the first technician will be the first to find the machine broken if and only if the first failure occurs in an odd run. We are given that \( X \sim \text{Geometric}(p) \) and hence

\[
q = \sum_{k=0}^{\infty} \mathbb{P}[X = 2k + 1] = \sum_{k=0}^{\infty} p(1-p)^{2k} = p \sum_{k=0}^{\infty} (1-2p+p^2)^k = \frac{p}{2p-p^2} = \frac{1}{2-p}.
\]

Alternatively, suppose we decompose the above sum as follows:

\[
q = \mathbb{P}[X = 1] + \sum_{i=1}^{\infty} \mathbb{P}[\{X = 2i + 1 \cap X \neq 1\}] = \mathbb{P}[X = 1] + \sum_{i=1}^{\infty} \mathbb{P}[X \neq 1] \mathbb{P}[X = 2i + 1 | X \neq 1]
\]
Using the memoryless property of the geometric distribution we have
\[ P[X = 2i + 1 | X \neq 1] = P[X = 2i + 1 | X > 1] = P[X = 2i], \]
and this further simplifies the above sum as follows:

\[
q = \frac{P[X = 1] + \sum_{i=1}^{\infty} P[X \neq 1] P[X = 2i]}{P[X = 1] + \sum_{i=1}^{\infty} P[X = 2i]}
\]

which is equal to:
\[
q = \frac{P[X = 1] + \sum_{i=1}^{\infty} P[X = 2i]}{P[X = 1] + \sum_{i=1}^{\infty} P[X = 2i - 1]}
\]

where the second last equality follows because the probability that \( X \) is even is the complement of the event that \( X \) is odd. The final equation is intuitive as in the event that the first technician doesn’t find the machine broken in the first run, the memoryless property of the geometric distribution ensures that the probability that the second technician finds the machine broken first is the same as the probability that the first technician does when we have no knowledge of the first run. That is, we have:

\[
q = P[\text{Second technician finds the machines broken first} \mid \text{No machine fails in run 1}].
\]

Using \( P[X = 1] = p \) and solving the above equation, we get that:

\[
q = \frac{1}{2 - p}.
\]

7 Exploring the Geometric Distribution

(a) Let \( X, Y \) be i.i.d. geometric random variables with parameter \( p \). Let \( U = \min\{X, Y\} \) and \( V = \max\{X, Y\} - \min\{X, Y\} \). Compute the joint distribution of \( (U, V) \)

(b) Prove that \( U \) and \( V \) are independent.

Solution:

(a) One has, for \( u, v \in \mathbb{N}, u, v \geq 1 \):

\[
P[U = u, V = v] = P[\min\{X, Y\} = u, \max\{X, Y\} = u + v] = P[X = u, Y = u + v] = P[X = u, Y = u + v] + P[X = u + v, Y = u]
\]

\[
= P[X = u] P[Y = u + v] + P[X = u + v] P[Y = u]
\]

\[
= p(1 - p)^{u-1} p(1 - p)^{u+v-1} + p(1 - p)^{u+v-1} p(1 - p)^{u-1} = 2p^2(1 - p)^{2u+v-2}.
\]
Also, for $u \in \mathbb{N}, u \geq 1$:
\[
P[U = u, V = 0] = P[X = Y = u] = P[X = u]P[Y = u] = p(1 - p)^{u-1} p(1 - p)^{u-1} = p^2(1 - p)^{2u-2}.
\]

Putting it together, we have:
\[
P[U = u, V = v] = \begin{cases} 
2p^2(1 - p)^{2u+v-2} & u, v \in \mathbb{N}, u \geq 1, v \geq 1 \\
p^2(1 - p)^{2u-2} & u \in \mathbb{N}, u \geq 1, v = 0 \\
0 & \text{otherwise}
\end{cases}
\]

(b) Now, to show that $U$ and $V$ are independent, we must compute their marginal distributions.
We have, for $u \in \mathbb{N}, u \geq 1$,
\[
P[U = u] = \sum_{v=0}^{\infty} P[U = u, V = v] = p^2(1 - p)^{2u-2} + \sum_{v=1}^{\infty} 2p^2(1 - p)^{2u+v-2}
= p^2(1 - p)^{2u-2} + 2p^2(1 - p)^{2u-1} p^{-1}
= (1 - p)^{2u-1} (p^2 + 2p - 2p^2)
= (2p - p^2)(1 - 2p + p^2)^{u-1}.
\]

Note that $U \sim \text{Geometric}(2p - p^2)$.

We have, for $v \in \mathbb{N}, v \geq 1$,
\[
P[V = v] = \sum_{u=1}^{\infty} P[U = u, V = v] = \sum_{u=1}^{\infty} 2p^2(1 - p)^{2u+v-2}
= 2p^2(1 - p)^v \frac{1}{2p - p^2} = \frac{2p(1 - p)^v}{2 - p}.
\]

And for $v = 0$,
\[
P[V = 0] = \sum_{u=1}^{\infty} P[U = u, V = 0] = \sum_{u=1}^{\infty} p^2(1 - p)^{2u-2}
= \frac{p^2}{2p - p^2} = \frac{p}{2 - p}.
\]

It is easily verified that
\[
P(U = u, V = v) = P(U = u)P(V = v) \quad \forall u, v \in \mathbb{N}, u \geq 1.
\]

so $U$ and $V$ are independent.
Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model $X$, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean $\lambda$.

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability $p$. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as $Y$ and the number of them that do not buy anything as $Z$ (so $X = Y + Z$).

(a) What is the probability that $Y = k$ for a given $k$? How about $\Pr[Z = k]$? *Hint:* You can use the identity $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(b) State the name and parameters of the distribution of $Y$ and $Z$.

(c) Prove that $Y$ and $Z$ are independent.

**Solution:**

(a) We consider all possible ways that the event $Y = k$ might happen: namely, $k + j$ people enter the store ($X = k + j$) and then exactly $k$ of them choose to buy something. That is,

$$\Pr[Y = k] = \sum_{j=0}^{\infty} \Pr[X = k + j] \cdot \Pr[Y = k \mid X = k + j]$$

$$= \sum_{j=0}^{\infty} \left( \frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \right) \cdot \left( \frac{(k+j)!}{k!} p^j (1-p)^k \right)$$

$$= \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} \cdot \frac{(k+j)!}{k!} \cdot \frac{p^j (1-p)^k}{k!} = \frac{(\lambda (1-p))^k e^{-\lambda}}{k!} \cdot \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{j!}$$

$$= \frac{(\lambda (1-p))^k e^{-\lambda}}{k!} \cdot e^{\lambda p} = \frac{\lambda (1-p))^k e^{-\lambda (1-p)}}{k!}.$$

The case for $Z$ is completely analogous:

$$\Pr[Z = k] = \frac{(\lambda p)^k e^{-\lambda p}}{k!}$$

(b) $Y$ follows the Poisson distribution with parameter $\lambda (1-p)$ and $Z$ follows the Poisson distribution with parameter $\lambda p$. 

CS 70, Fall 2019, HW 12
(c) The joint distribution of $Y$ and $Z$ is given by

$$
P(Y = y, Z = z) = \sum_{x=0}^{\infty} P(X = x, Y = y, Z = z) = \sum_{x=0}^{\infty} P(Y = y, Z = z \mid X = x) P(X = x)
$$

$$
= P(Y = y, Z = z \mid X = y + z) P(X = y + z) = \frac{(y+z)!}{y!z!} p^y (1-p)^z \frac{e^{-\lambda} \lambda^{y+z}}{(y+z)!}
$$

$$
= \frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \cdot \frac{e^{-\lambda p (\lambda p)^z}}{z!} = P(Y = y) \cdot P(Z = z).
$$

Since $P(Y = y, Z = z) = P(Y = y) \cdot P(Z = z)$ for all $y, z \in \mathbb{N}$, we get that $Y$ and $Z$ are independent.