CS 70 Discrete Mathematics and Probability Theory
Spring 2020 Course Notes

HW 7

1 Stirling's Approximation

In this question, suppose $n \in \mathbb{Z}^+$, we want to find approximations for $n!$. To plot the given functions, you can use an online tool (*e.g.*, go to http://www.wolframalpha.com/ and type "plot $\ln x$").

1. Plot the function $f(x) = \ln x$.
2. For the following three questions, please note that $\ln x$ is strictly increasing and concave-$\cap$ because, when $x > 0$, its first and second derivatives are positive and negative, respectively. Concavity means that all line segments connecting two points on the function are below the function.
   Suppose $n \in \mathbb{Z}^+$, use the plot to explain why
   $$\ln 1 + \ln 2 + \ldots + \ln n \geq \int_1^n \ln x dx \tag{1}$$
3. Suppose $n \in \mathbb{Z}^+$, use the plot to explain why
   $$\ln 1 + \ln 2 + \ldots + \ln n < \int_1^{n+1} \ln x dx \tag{2}$$
4. Suppose $a \in \mathbb{Z}^+$, use the plot to explain why
   $$\left(\frac{\ln a + \ln(a+1)}{2}\right) < \int_a^{a+1} \ln x dx \tag{3}$$
5. Use Equation ((1)) to prove $n! \geq e\left(\frac{n}{e}\right)^n$.
6. Use Equation ((2)) to prove $n! \leq en\left(\frac{n}{e}\right)^n$ (Hint: If in this part you find yourself wishing you had $n-1!$ on the left-hand-side, try to prove an upper bound on $n-1!$ and use that to help you)
7. Use Equation ((3)) to prove $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$, which is a tighter upper bound.
8. The Stirling's approximation is usually written as $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ or a simpler version $n! \approx \left(\frac{n}{e}\right)^n$. Plot the function $f(n) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!}$. What do you observe?
9. Suppose $m = \frac{k}{n}$, use m, n and apply the simpler version of the Stirling's approximation to rewrite $\binom{n}{k}$.

10. Now, suppose $m_1 = \frac{k_1}{n} = 0.25$, $m_2 = \frac{k_2}{n} = 0.5$, and $m_3 = \frac{k_3}{n} = 0.75$, plot $\ln(\binom{n}{k_1})$, $\ln(\binom{n}{k_2})$, and $\ln(\binom{n}{k_3})$ as functions of n on a plot with linear-scaled axes. What do you observe?

Solution:

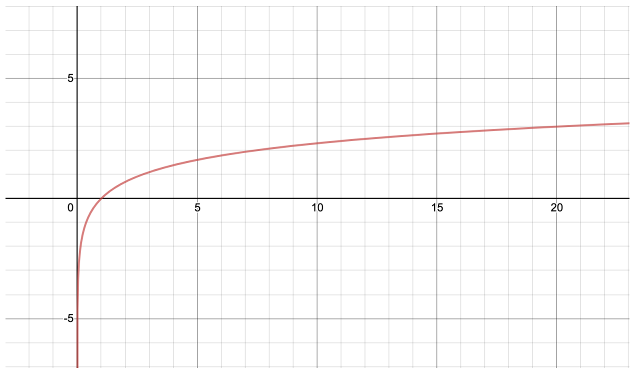

Figure 1: The plot of $\ln x$. Source: desmos.com

(a)

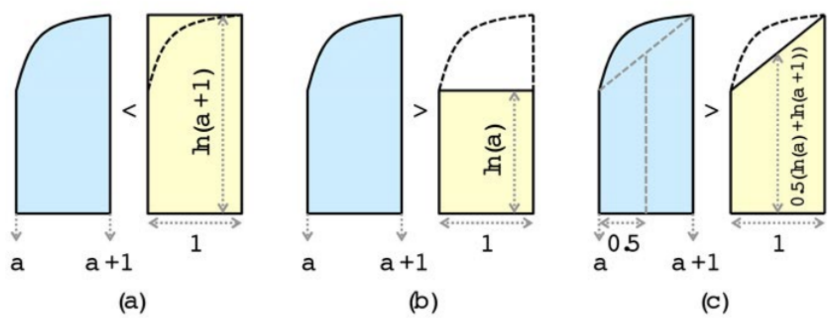

Figure 2: The area comparisons.

(b) If $n = 1$, both sides are equal to 0. If $n \geq 2$, since $\ln x$ is strictly increasing and concave-$\cap$, we can see from Figure 2 (a) that the area of the left colored region is smaller than the right colored region, *i.e.*, $\int_a^{a+1} \ln x dx < \ln(a+1)$ where $a \geq 1$, so

$$\begin{aligned} \ln 1 + \ln 2 + \ldots + \ln n &= 0 + \ln 2 + \ln 3 + \ldots + \ln n \\ &> \int_1^2 \ln x dx + \int_2^3 \ln x dx + \ldots + \int_{n-1}^n \ln x dx \\ &= \int_1^n \ln x dx. \end{aligned}$$

(c) Since $\ln x$ is strictly increasing and concave, we can see from Figure 2 (b) that the area of the left colored region is larger than the right colored region, *i.e.*, $\int_a^{a+1} \ln x dx > \ln a$ where $a \geq 1$, so

$$\begin{aligned}\ln 1 + \ln 2 + \ldots + \ln n &< \int_1^2 \ln x dx + \int_2^3 \ln x dx + \ldots + \int_n^{n+1} \ln x dx \\ &= \int_1^{n+1} \ln x dx.\end{aligned}$$

(d) Since $\ln x$ is strictly increasing and concave, we can see from Figure 2 (c) that the area of the left colored region is larger than the right colored region, *i.e.*, $\int_a^{a+1} \ln x dx > \left(\frac{\ln a + \ln(a+1)}{2}\right)$ where $a \geq 1$.

(e) We have

$$\begin{aligned}\ln(n!) &= \ln 1 + \ln 2 + \ldots + \ln n \\ &\geq \int_1^n \ln x dx \\ &= (x \ln x - x)|_1^n \\ &= (n \ln n - n) - (0 - 1) \\ &= n \ln n - n + 1 \\ &= \ln(n^n) - \ln(e^n) + \ln e \\ &= \ln\left(e\left(\frac{n}{e}\right)^n\right),\end{aligned}$$

so $n! \geq e\left(\frac{n}{e}\right)^n$.

(f) If $n = 1$, $n! = en\left(\frac{n}{e}\right)^n = 1$. If $n > 1$, we have

$$\begin{aligned}\ln(n!) &= \ln 1 + \ln 2 + \ldots + \ln n \\ &= (\ln 1 + \ln 2 + \ldots + \ln(n-1)) + \ln n \\ &< \int_1^n \ln x dx + \ln n \\ &= (x \ln x - x)|_1^n + \ln n \\ &= (n \ln n - n) - (0 - 1) + \ln n \\ &= n \ln n - n + 1 + \ln n \\ &= \ln(n^n) - \ln(e^n) + \ln e + \ln n \\ &= \ln\left(en\left(\frac{n}{e}\right)^n\right),\end{aligned}$$

so $n! < en\left(\frac{n}{e}\right)^n$ for $n > 1$, and the claim $n! \leq en\left(\frac{n}{e}\right)^n$ is proved.

(g) If $n=1$, $n! = e\sqrt{n}\left(\frac{n}{e}\right)^n = 1$. If $n>1$, we have

$$\begin{aligned}
\ln(n!) &= \ln 1 + \ln 2 + \ldots + \ln n \\
&= \frac{\ln 1}{2} + \left(\frac{\ln 1 + \ln 2}{2}\right) + \left(\frac{\ln 2 + \ln 3}{2}\right) + \ldots + \left(\frac{\ln(n-1) + \ln n}{2}\right) + \frac{\ln n}{2} \\
&< 0 + \int_1^2 \ln x dx + \int_2^3 \ln x dx + \ldots + \int_{n-1}^n \ln x dx + \frac{\ln n}{2} \\
&= \int_1^n \ln x dx + \frac{\ln n}{2} \\
&= (x \ln x - x)|_1^n + \frac{\ln n}{2} \\
&= (n \ln n - n) - (0-1) + \frac{\ln n}{2} \\
&= n \ln n - n + 1 + \frac{\ln n}{2} \\
&= \ln(n^n) - \ln(e^n) + \ln e + \ln(\sqrt{n}) \\
&= \ln\left(e\sqrt{n}\left(\frac{n}{e}\right)^n\right),
\end{aligned}$$

so $n! < e\sqrt{n}\left(\frac{n}{e}\right)^n$ for $n>1$, and the claim $n! \le e\sqrt{n}\left(\frac{n}{e}\right)^n$ is proved.

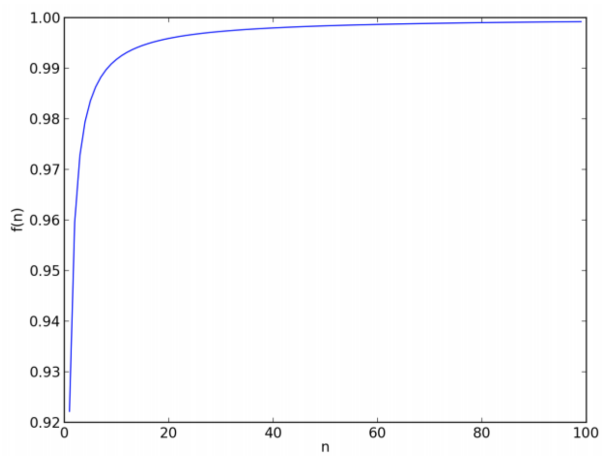

Figure 3: The plot of the ratio which is $f(n) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!}$.

(h) The plot is shown in Figure 3. The function is closer to 1 as n increases.

(i) We have

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{(n-k)!k!} \\
&\approx \left(\frac{n^n}{e^n}\right)\left(\frac{e^{n-k}}{(n-k)^{n-k}}\right)\left(\frac{e^k}{k^k}\right) \\
&= \frac{n^n}{(n-k)^{n-k}k^k} \\
&= \left(\frac{n}{n-k}\right)^{n-k}\left(\frac{n}{k}\right)^k \\
&= \left(\frac{1}{1-m}\right)^{(1-m)n}\left(\frac{1}{m}\right)^{mn}.
\end{aligned}$$

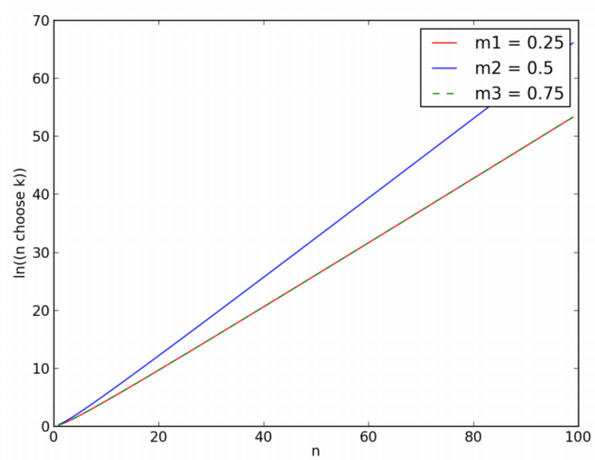

Figure 4: The plot of approximated $\ln(\binom{n}{k_i})$ where $m_1 = 0.25$, $m_2 = 0.5$, and $m_3 = 0.75$.

(j) The plot is shown in Figure 4. The functions plotted are nearly linear! The functions with $m_1 = 0.25$ and $m_3 = 0.75$ overlap with each other.

2 Double Injections

The zero-th law of counting says that if a bijection (one-to-one and onto mapping) exists between two sets A and B, the two sets have the same size. In some cases, rather than coming up with a single bijection, it is easier to come up with two different one-to-one mappings. One of them going from A to B, and the other going from B to A.

The spirit of the pigeon-hole principle intuitively tells us that if we can find an assignment of pigeons to holes such that no two pigeons are in the same hole and every pigeon has a hole, as well as a possibly distinct assignment of holes to pigeons so that no two holes are assigned to the same pigeon and every hole has a pigeon, then there must be the same number of holes as pigeons. But this intuitive fact actually needs a proof. (Especially if we want to consider infinite sets.) This fact is called the Cantor-Schröder-Bernstein theorem, and this problem walks you through the proof of it.

Assume that there exist injective (one-to-one) functions $f : A \to B$ and $g : B \to A$.

The key here is to somehow construct a bijection h from these two maps. Our goal is to reuse the functions f and g as much as possible, but we have to confront the key question of when do we use f and when do we use g.

To navigate this key question, we define the *chain* of an element $a \in A$ as the following sequence:

$$\cdots \to f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to \cdots$$

And similarly, the chain of an element $b \in B$ as:

$$\cdots \to g^{-1}(f^{-1}(b)) \to f^{-1}(b) \to b \to g(b) \to f(g(b)) \to \cdots$$

We will use the notation $C(a)$ to denote the chain of a. (And similarly for b.) Note that $C(a)$ will always extend infinitely to the right of a (i.e. the sequence $a, f(a), g(f(a)), \cdots$ does not terminate), because we can always apply f and g to elements in A and B, respectively. However, the same is not true when extending to the left of a, because the inverse mappings g^{-1} and f^{-1} may not exist for certain elements in A or B. After all, neither of the mappings f and g are promised to be onto mappings. They are only guaranteed to be one-to-one.

It turns out that there are four types of chains:

1. **Cyclic Chains** form a loop. For example, the sequence $a_1 \to b_1 \to a_2 \to b_2 \to a_1$ is cyclic.
2. **Doubly Infinite Chains** extend infinitely in the leftwards direction.
3. **A-Stoppers** end (on the left) in A. That is, an $f^{-1}(\cdot)$ lands us in an element of A that isn't in the range of g.
4. **B-Stoppers** end (on the left) in B. That is, a $g^{-1}(\cdot)$ lands us in an element of B that isn't in the range of f.

(a) Our definition of a cyclic chain above is quite restrictive — it requires the entire chain to form a single loop. Indeed, if there is ever a cycle in a chain, the entire chain is always a single loop. **Argue why this is always the case.**

(HINT: For example, the sequence $a_1 \to b_1 \to a_2 \to b_2 \to a_3 \to b_1$, where all elements are distinct, is not a valid chain. Why?)

(b) Two chains are distinct if one contains any element that the other one does not. They are the same chain if they contain all the same elements. **Prove that every element of A and B is part of exactly one chain.** Note that many elements can share the same chain (e.g. a and $f(a)$ are always in the same chain).

(HINT: You need to show that if two chains intersect at all, they must be the same chain.)

(c) We will now try to demonstrate that a bijection $h : A \to B$ exists. Since every element of A and B is part of exactly one chain, we will try to demonstrate that h will be a bijection for chains of a particular type. That is, to prove that h is a bijection for chains of type T, we will demonstrate that, for all chains C of type T, the function h is a bijection from C_A (the subset of A in the chain C) to C_B (the subset of B in the chain C).

Your friend suggests using f as a candidate bijection (i.e. $h(x) = f(x)$). **For which types of chains will f be a bijection? For which types will f not be a bijection? Justify your answer, for each case: prove that f is a bijection, or argue why f need not be a bijection.**

(d) In the previous part, you will have noticed that f is not necessarily a bijection when a belongs to a B-stopper. **Demonstrate that for all chains C that are B-Stoppers, g^{-1} exists and is a bijection from C_A to C_B.**

(e) Using what you've learned in the previous parts, you sugggest the following function as a candidate bijection:

$$h(x) = \begin{cases} g^{-1}(a) \text{ ,if } a \text{ is part of a B-Stopper} \\ f(a) \text{ ,otherwise} \end{cases}$$

Demonstrate that h is now a bijection between A and B.

(*Hint: Use your analysis from the previous parts.*)

Solution:

(a) Looking at the example sequence given, you may notice that two arrows are pointing to element b_1: a_1, and a_3. This is a problem, because it implies that $f(a_1) = b_1$, and $f(a_3) = b_1$. Assuming that a_1 and a_3 are distinct, this violates the one-to-one condition, since $f(a_1) = f(a_3)$, while $a_1 \neq a_3$.

We generalize this argument. Suppose that a chain forms a cycle, but does not form a complete loop. This implies that some element, e, must be connected to an element that is outside of the cycle. Without loss of generality, suppose that $e \in A$. We observe that e can not point to a separate element outside of the cycle; otherwise, $f(e)$ would map to two distinct elements in B. Furthermore, an element outside of the cycle can not point to e; otherwise, g would map two elements in B to e, which contradicts the one-to-one condition of f.

(b) Each element a is in at least one chain: it's own chain $C(a)$.

We now prove that an element can not be in more than one chain. Inspecting the chain given in the problem statement, we observe that a is always followed by $f(a)$; this should be true in every chain involving a, since we must extend the chain by applying f. Extending this logic, $g(f(a))$ must always follow, too. In general, we apply one of f, g, f^{-1} or g^{-1} to move along the chain. As a result, it's difficult to imagine the existence of two different chains involving a, since the given functions map deterministically to a single element (or, to nothing).

We formalize by contradiction. Assume that some element $a \in A$ appears in two distinct chains. However, by definition, all elements to the right of a must be the same in both sequences âĂŞ otherwise, $f(a')$ would map to two distinct elements in B, for some $a' \in A$, or vice-versa, which violates the definition of a function. Furthermore, all elements to the left of a must be the same in both sequences, for the same reason. Thus, the sequences must be identical, which contradicts the original assumption. An identical argument can be applied for elements in B.

(c) h is a bijection for elements in Cyclic Chains, Doubly Infinite Chains, and and A-Stoppers. It is given that f is an injection from A to B; thus, it suffices to demonstrate that f is onto for elements in such chains.

We proceed by demonstrating that h is onto for elements of the types of chains listed above. Given an arbitrary $b \in B$, we must demonstrate that there exists an $a \in A$ such that $h(a) = b$. If b is in a chain of a Cyclic Chain, Doubly Infinite Chain, or an A-Stopper, then there is always some element in the chain that extends to the left of b; thus, there is always some element in A that maps to b.

However, if b is part of a B-Stopper, this is not always the case. In particular, consider the element that the chain terminates on; call it t. Since the chain is a B-stopper, t must be a part of B. This implies that an inverse mapping f^{-1} does not exist for t, which means that there is no element in A such that $h(a) = t$. Thus, h is not onto for elements in B-Stoppers.

(d) We demonstrate that $g^{-1}(a)$ is a bijection for elements that are a part of a B-Stopper.

First, we demonstrate that g^{-1} is one to one between C_A and C_B; that is, if $g^{-1}(a_1) = g^{-1}(a_2)$, then $a_1 = a_2$. Suppose that $g^{-1}(a_1) = b_1$ and $g^{-1}(a_2) = b_2$; furthermore, suppose that $g(b_1) = a_1$, and $g(b_2) = a_2$. Since $g^{-1}(a_1) = g^{-1}(a_2)$, we have that $b_1 = b_2$. Thus, $g(b_1) = g(b_2)$, which implies that $a_1 = a_2$.

Now, we demonstrate that g^{-1} is onto C_B, for any chain C. If b is in the chain, this implies that $g(b)$ is in the chain. Note that $g(b)$ is an element of A; furthermore, observe that $g^{-1}(g(b)) = b$; which means that $g(b)$ maps to b. Thus, g^{-1} is onto (and therefore bijective) for elements in B-stoppers.

(e) Observe that h is well-defined since every element of A appears in exactly one chain.

Moreover, h is onto since every $b \in B$ appears in exactly one chain C, and there exists a function (either f or g^{-1}) from C_A to C_B for which it is onto.

It sufficies to show that $h(x)$ is one-to-one. We demonstrate this by cases: *Elements in different chains*: Note that if $a \in C$ and $a \in C'$, for $C \neq C'$, then $h(a)$ and $h(a')$ must be a part of different chains; thus, $h(a) \neq h(a')$. *Elements in the same chain*: If $a, a' \in C$, we use the same function (either f or g^{-1}) to map to B, for both elements. In the previous parts, we have demonstrated that this is an injective mapping; thus, in this case, h will also be injective.

This concludes the proof. This theorem is known as the Schroder-Bernstein therorem (it is sometimes called the Cantor-SchroderâĂŞBernstein theorem after Georg Cantor, who initially published the theorem without proof).

3 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(k) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(l) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(m) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

(a) $\binom{n+k}{k}$

(b) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6}\binom{39}{7}$ ways to make up the hand.

(c) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $104!/2^{52}$.

(d) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.
This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$$

by substituting $l = 99-k$. Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = (1/2)\cdot 2^{99} = 2^{98}$.

Answer 2: Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = (1/2)\cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(e) This is the number of ways of rearranging 7 distinct letters and is 7!.

In this 6 letter word, the letter A is repeated 3 times while the other letters appear once. Hence, the number 6! overcounts the number of different anagrams by a factor of 3! (which is the number of ways of permuting the 3 A's among themselves). Hence, there are $6!/3!$ different anagrams.

In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of 4! (which is the number of ways of permuting the 4 A's among themselves). Hence, there are $7!/4!$ anagrams.

In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number 7! overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of 2! for the number of ways of permuting the 2 A's among themselves and another factor of 2! for the number of ways of permuting the 2 N's among themselves). Hence, there are $7!/(2!)^2$ different anagrams.

(f) (1) We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is 5!. (2) Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \bigcup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = 6!/2$.

(g) Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.

(h) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:
Case 1: The 2 balls land in the same bin. This gives 7 ways.
Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.
Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. From class (see notes 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 7$, we get $\binom{8}{2}$ ways to do this.
EASY EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 9$ and $n = 27$, so there are $\binom{35}{9}$ ways.

(j) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x,y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $20!/(10! \cdot 2^{10})$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\dots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\dots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

(k) $\binom{n+k}{k}$. There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has.

(l) $n-1$. It's easy just to enumerate the solutions here. x_0 can take values $1,2,\dots,n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(m) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

4 Shipping Crates

A widget factory has four loading docks for storing crates of ready-to-ship widgets. Suppose the factory produces 8 indistinguishable crates of widgets and sends each crate to one of the four loading docks.

(a) How many ways are there to distribute the crates among the loading docks?

(b) Now, assume that any time a loading dock contains at least 5 crates, a truck picks up 5 crates from that dock and ships them away. (e.g., if 6 crates are sent to a loading dock, the truck removes 5, leaving 1 leftover crate still in the dock). We will now consider two configurations to be identical if, for every loading dock, the two confirgurations have the same number of leftover crates in that dock. How would your answer in the previous part compare to the number of outcomes given the new setup? Justify your answer.

(c) We will now attempt to count the number of configurations of crates. First, we look at the case where crates are removed from the dock. How many ways are there to distribute the crates such that some crate gets removed from the dock?

(d) How many ways are there to distribute the crates such that no crates are removed from the dock; i.e. no dock receives at least 5 crates?

(e) Putting it together now, what are the total number of possible configurations for crates in the modified scenario? *Hint:* Observe that, regardless of which dock receives the 5 crates, we end up in the same situation.

After all the shipping has been done, how many possible configurations of leftover crates in loading docks are there?

Solution:

(a) This can be solved using stars and bars. We are simply distributing 8 indistinguishable balls into 4 distinguishable bins; the total amount of ways to count this is $\binom{11}{3}$.

(b) There are less possible outcomes in the new setup. The effect of the truck is that of a function, mapping the configurations that we counted in the previous part to a new set of outcomes; although the function may map two distinct configurations to the same outcome, it will certainly not map the same configuration to two different new outcomes. Thus, $\binom{11}{3}$ is a valid upper bound, which indicates that the number of outcomes is still finite (which means we can count it)!

(c) You may notice that it's only possible for one truck to receive five crates; we will leverage this fact in order to simplify our counting. We will count the number of ways to distribute the crate such that some dock receives 5 crates. Observe that, regardless of which dock receives the 5 crates, the scenario reduces to the same thing: we are simply distributing the leftover 3 crates among 4 docks. There are 4 ways to choose which dock has ≥ 5 crates, and $\binom{6}{3}$ ways to distribute the leftover 3 crates. Thus, there are $4\binom{6}{3}$ ways to distribute the crates such that some dock receives 5 crates.

(d) You can put all outcomes into one of two categories: outcomes in which some dock receives more than five crates, and outcomes in which no dock receives more than five crates. Note that these categories are mutually exclusive. As a result, we can take the complement of the previous part; the number of outcomes in which no crates are removed from the dock is simply the total number of outcomes, subtracted by the number of outcomes in which some dock receives more than five crates (found in the previous part). Putting it together, this gives us $\binom{11}{3} - 4\binom{6}{3}$.

(e) Regardless of which dock receives the 5 crates, we are left distributing 3 indistinguishable crates amount 4 docks. Using stars and bars, the total number of outcomes is $\binom{6}{3}$. Thus, to obtain the total amount of configurations, we count the amount of outcomes without removal of the crates, and add this to the amount of outcomes after the removal of the crates. Putting it together, we have that the total number of outcomes is

$$\binom{11}{3} - 4\binom{6}{3} + \binom{6}{3} = \binom{11}{3} - 3\binom{6}{3} = 105.$$

5 Binomial Theorem

Imagine you are throwing n numbered balls into bins. There are x red bins and y blue bins.

Use the above scenario in a combinatorial argument to prove the binomial theorem, which states the following:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Solution:

LHS: This is the number of ways of throwing n distinct balls into the bins.

RHS: Consider each term $\binom{n}{k}x^{n-k}y^k$. This is the number of ways of throwing the balls with k landing in the blue bins and $n-k$ landing in the red bins. The choose gives us the number of ways of dividing the balls between landing in blue and landing in red. The value x^{n-k} gives us the number of ways that $n-k$ balls can be thrown into the red bins. The value y^k gives us the number of ways that k balls can be thrown into the blue bins. Therefore for each way of allocating k balls to blue and rest to red, we have x^{n-k} ways for them to land in red and y^k ways for them to land into blue. Therefore the product gives the number of ways to throw the balls given that k land in the blue bins and $n-k$ land in the red bins.

If we sum up the terms on the right hand side, we get the total number of ways of throwing the balls into the bins.

6 Charm School Applications

(a) n males and n females apply to the Elegant Etiquette Charm School (EECS) within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

(b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The boy(s) get in one car, and the girl(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Each car has a driver...)

Solution:

(a) One way of counting is simply $\binom{2n}{n}$, since we must pick n students from $2n$.

The other way is to first pick i males, then $n-i$ females. Equivalently, choose i males to admit, and i females to NOT admit. For a fixed i, this yields $\binom{n}{i}\binom{n}{n-i} = \binom{n}{i}^2$ choices. Thus, over all choices of i:

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

(b) Out of the n males and n females who applied, count the number of ways that accepted students can drive to school.

RHS: First pick one male driver and one female driver from the n male and n female applicants (n^2). Then pick the other $n-2$ accepted students from the pool of $2n-2$ remaining applicants.

LHS: Pick k males and $n-k$ females that were accepted: $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. Then pick a driver among the k males, and among the $n-k$ females. Because the problem statement says there is at least 1 girl and 1 boy, k can range from 1 to $n-1$.

7 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

(a) k^p. For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.

(b) $k^p - k$. You can have k sequences of a beads with only one color.

(c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p-1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.

(d) $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p, i.e., $k^p \equiv k \pmod{p}$, which is Fermat's Little Theorem.

8 Make Your Own Question

You must make your own question on this week's material and solve it.

9 Homework Process and Study Group

You must describe your homework process and study group in order to receive credit for this question.