1 Maze

Let’s assume that Tom is located at the bottom left corner of the $9 \times 9$ maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

(a) How many such shortest paths exist?

(b) How many shortest paths pass through the edge labeled $X$? The edge labeled $Y$? Both the edges $X$ and $Y$? Neither edge $X$ nor edge $Y$?

(c) How many shortest paths pass through the vertex labeled $Z$? The vertex labeled $W$? Both the vertices $Z$ and $W$? Neither vertex $Z$ nor vertex $W$?

Solution:

(a) Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let’s call these “H” edges) and 9 vertical edges going up (let’s call these “V” edges).

Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 “H”s and 9 “V”s. For example, one such path is $\text{HHHHHHHH-HVVVVVVVV}$ (which represents the path that goes all the way to the right, and then all
the way to the top). Conversely, every such sequence of exactly 9 “H”s and 9 “V”s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 “H”s and 9 “V”s in a sequence, which is \( \binom{18}{9} = 48620 \).

(b) For a shortest path to pass through the edge \( X \), it has to first get to the left vertex of \( X \). So the first portion of the path has to start at the origin, and end at the left vertex of \( X \). Using the same logic as above, there are exactly \( \binom{6}{3} = 20 \) ways to complete this “first leg” of the path (consisting of 3 “H” edges and 3 “V” edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of \( X \) to the top right corner of the maze (the “second leg”). This second leg will consist of 5 “H” edges and 6 “V” edges, and using the same logic, there are exactly \( \binom{11}{5} = 462 \) possibilities. Therefore, the total number of shortest paths that pass through the edge \( X \) is \( 20 \times 462 = 9240 \).

Using similar logic, any shortest path that passes through \( Y \) has to consist of 2 legs, the first leg going from the origin to the bottom vertex of \( Y \), and the second leg going from the top vertex of \( Y \) to the top right corner of the maze. The first leg will consist of exactly 5 “H”s and 4 “V”s, while the second leg will consist of exactly 4 “H”s and 4 “V”s. So the total number of such shortest paths will be \( \binom{9}{5} \times \binom{8}{4} = 8820 \).

By a similar argument, let’s try to figure out how many paths will pass through both \( X \) and \( Y \). Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 “H”s and 3 “V”s (going from the origin to the left edge of \( X \), the second leg consisting of 1 “H” and 1 “V” (going from the right vertex of \( X \) to the bottom vertex of \( Y \)), and the third leg consisting of 4 “H”s and 4 “V”s (going from the top vertex of \( Y \) to the top right corner of the maze). The total number of such shortest paths is therefore \( \binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800 \).

Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through \( X \), 8820 pass through \( Y \), and 2800 pass through both. So the number of paths that pass through neither
is 33360 (see the figure above for an intuitive explanation).

(c) This part is very similar in spirit to the previous one, except that in this case, each leg of the path we consider begins exactly where the previous leg ended, and not to the right or to the top of where the previous leg ended.

For concreteness, let’s find out how many shortest paths pass through vertex Z. Observe that for a shortest path to pass through Z, it has to first get to Z. So the first portion of the path has to start at the origin, and end at Z. Using the same logic as above, there are exactly \( \binom{11}{4} = 330 \) ways to complete this “first leg” of the path (consisting of 4 “H” edges and 7 “V” edges). Having chosen one of these 330 ways, the path has to then go from Z to the top right corner of the maze. This second leg will consist of 5 “H” edges and 2 “V” edges, and so there are exactly \( \binom{7}{2} = 21 \) possibilities. Therefore, the total number of shortest paths that pass through the vertex Z is \( 330 \times 21 = 6930 \).

Using similar logic, any shortest path that passes through W has to consist of 2 legs, the first leg going from the origin to W, and the second leg going from W to the top right corner of the maze. The first leg will consist of exactly 7 “H”s and 8 “V”s, while the second leg will consist of exactly 2 “H”s and 1 “V”. So the total number of such shortest paths will be \( \binom{15}{7} \times \binom{3}{1} = 19305 \).

By a similar argument, let’s try to figure out how many paths will pass through both Z and W. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 4 “H”s and 7 “V”s (going from the origin to Z), the second leg consisting of 3 “H”s and 1 “V” (going from Z to W), and the third leg consisting of 2 “H”s and 1 “V” (going from W to the top right corner of the maze). The total number of such shortest paths is therefore \( \binom{11}{4} \times \binom{4}{1} \times \binom{3}{1} = 3960 \).

Finally, we know that there are 48620 shortest paths in all, of which 6930 pass through Z, 19305 pass through W, and 3960 pass through both. So the number of paths that pass through neither is 26345 (see the figure above for an intuitive explanation).
2 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) \( \binom{n}{i} = \binom{n}{n-i} \).

(b) \( \sum_{i=1}^{n} i \binom{n}{i}^2 = n \binom{2n-1}{n-1} \). (Hint: Part (a) might be useful.)

(c) \( \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n \). (Hint: consider the number of ways of splitting \( n \) elements into 3 groups.)

Solution:

(a) Choosing \( i \) players out of \( n \) to play on a team is the same as choosing \( n-i \) players to not play on the team, i.e. \( \binom{n}{i} = \binom{n}{n-i} \).

(b) Assume we have \( n \) women and \( n \) men. Using part (a) we can rewrite the LHS as \( \sum_{i=1}^{n} i \binom{n}{i} \binom{n}{n-i} \), which we can interpret as selecting a team of \( n \) players by choosing \( i \) women and \( n-i \) men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining \( n-1 \) players from all remaining men and women to form the team.

(c) We count the number of ways to split \( n \) elements into 3 labeled groups by two different methods.

RHS: There are 3 different choices for each element, so \( 3^n \) for all of them.

LHS: For every \( i \) from 0 to \( n \), choose \( i \) elements to go in group A, then for every \( j \) from 0 to \( n-i \) choose \( j \) elements to go in group B, the remaining go in group C. This gives:

\[
\sum_{i=0}^{n} \binom{n}{i} \sum_{j=1}^{n-i} \binom{n-i}{j}
\]

3 Fibonacci Fashion

You have \( n \) accessories in your wardrobe, and you’d like to plan which ones to wear each day for the next \( t \) days. As a student of the Elegant Etiquette Charm School, you know it isn’t fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you’d like to plan which accessories to wear each day represented by subsets \( S_1, S_2, \ldots, S_t \), where \( S_1 \subseteq \{1,2,\ldots,n\} \) and for \( 2 \leq i \leq t \), \( S_i \subseteq \{1,2,\ldots,n\} \) and \( S_i \) is disjoint from \( S_{i-1} \).

(a) For \( t \geq 1 \), prove that there are \( F_{t+2} \) binary strings of length \( t \) with no consecutive zeros (assume the Fibonacci sequence starts with \( F_0 = 0 \) and \( F_1 = 1 \)).
(b) Use a combinatorial proof to prove the following identity, which, for \( t \geq 1 \) and \( n \geq 0 \), gives the number of ways you can create subsets of your \( n \) accessories for the next \( t \) days such that no accessory is worn two days in a row:

\[
\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_1-x_2}{x_3} \cdots \binom{n-x_1-x_2-\cdots-x_{t-1}}{x_t} = (F_{t+2})^n.
\]

(You may assume that \( \binom{a}{b} = 0 \) whenever \( a < b \).)

**Solution:**

(a) We will prove this by strong induction.

Base cases: For \( k = 1 \), the only binary strings possible are 0 and 1. Therefore, there are two possible binary strings, and \( F_{k+2} = F_3 = 2 \). For \( k = 2 \), the binary strings possible are 11, 01, and 10, and we have \( F_{k+2} = F_4 = 3 \), so the identity holds.

Inductive hypothesis: For \( k \geq 2 \), assume that for all \( 1 \leq x \leq k \), there are \( F_{x+2} \) binary strings of length \( x \) with no consecutive zeros.

Inductive step: Consider the set of binary strings of length \( k + 1 \) with no consecutive zeros. We can group these into two sets: those which end with 0, and those which end with 1. For those that end with a 0, these can be constructed by taking the set of binary strings of length \( k - 1 \) with no consecutive zeros and appending 10 to the end of them. Then by the inductive hypothesis, this set is of size \( F_{k+1} \). For those that end with a 1, these can be constructed by taking the set of binary strings of length \( k \) with no consecutive zeros and appending a 1 to the end of them. Then by the inductive hypothesis, this set is of size \( F_{k+1} \).

Since the union of these two subsets (those which end with 0 and those which end with 1) cover all possible elements in the set of binary strings of length \( k + 1 \) with no consecutive zeros, the size of this set will be \( F_{k+1} + F_{k+2} = F_{k+3} \). This thus proves the inductive hypothesis.

(b) We first consider the left-hand-side of the identity. To create subsets of accessories that are consecutively disjoint with sizes \( x_i = |S_i| \), \( 1 \leq i \leq n \), there are \( \binom{n}{x_1} \) ways to create \( S_1 \), the subset of accessories you will wear on the first day. Then since \( S_2 \) must be disjoint from \( S_1 \), there are \( \binom{n-x_1}{x_2} \) ways choose accessories to create \( S_2 \). Since \( S_3 \) must be disjoint from \( S_2 \), there are \( \binom{n-x_1-x_2}{x_3} \) ways choose accessories to create \( S_3 \), and so on. Thus there are \( \binom{n}{x_1} \binom{n-x_1}{x_2} \cdots \binom{n-x_1-x_2-\cdots-x_{t-1}}{x_t} \) ways to create subsets of accessories \( S_1, \ldots, S_t \) with respective sizes \( x_1, \ldots, x_t \). Then altogether, \( S_1, \ldots, S_t \) can be created in

\[
\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_1-x_2}{x_3} \cdots \binom{n-x_1-x_2-\cdots-x_{t-1}}{x_t}
\]

ways.

Now, consider the right-hand-side of the identity. Now for each accessory \( i \in \{1, \ldots, n\} \), we will first decide which subsets \( S_1, \ldots, S_t \) will contain accessory \( i \), where we can’t assign item
To consecutive subsets. For each accessory, we create a binary string of length \( t \), where the leading digit represents \( S_1 \), the next digit represents \( S_2 \), and so on. We will say that a 0 in digit \( k \) means that we will wear the accessory on day \( k \). Therefore, the number of ways we can assign accessory \( i \) to subsets \( S_1, \ldots, S_t \) such that no two consecutive subsets both have accessory \( i \) is the same as the number of binary strings of length \( t \) with no consecutive zeros. Thus using the result in part (a), there are \( F_t + 2 \) ways to select the nonconsecutive subsets containing \( i \) among \( S_1, \ldots, S_t \). Since we have \( n \) accessories, accessories 1, \ldots, \( n \) can be placed into subsets \( S_1, \ldots, S_t \) in \( (F_t + 2)^n \) ways. This thus proves the identity.

4 Probability Warm-Up

(a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls out of the bucket, what is the probability of getting exactly \( k \) red balls (assuming \( 0 \leq k \leq 20 \)) if the sampling is done with replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?

(b) Same as part (a), but the sampling is without replacement, i.e. after we take a ball out the bucket we do not return the ball back to the bucket.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

Solution:

(a) Let \( A \) be the event of getting exactly \( k \) red balls. Then treating all balls as distinguishable, we have a total of \( 100^{20} \) possibilities to draw a sequence of 20 balls. In order for this sequence to have exactly \( k \) red balls, we need to first assign them one of \( \binom{20}{k} \) possible locations within the sequence. Once done so, we have \( 30^k \) ways of actually choosing the red balls, and \( 70^{20-k} \) possibilities for choosing the blue balls. Thus in total we arrive at

\[
\mathbb{P}(A) = \frac{\binom{20}{k} \cdot 30^k \cdot 70^{20-k}}{100^{20}} = \binom{20}{k} \left( \frac{3}{10} \right)^k \left( \frac{7}{10} \right)^{20-k}.
\]

(b) We note that the size of the sample space is now \( \binom{100}{20} \), since we are choosing 20 balls out of a total of 100. To find \( |A| \), we need to be able to find out how many ways we can choose \( k \) red balls and \( 20 - k \) blue balls. So we have that \( |A| = \binom{30}{k} \left( \binom{70}{20-k} \right) \). So

\[
\mathbb{P}(A) = \frac{\binom{30}{k} \left( \binom{70}{20-k} \right)}{\binom{100}{20}}.
\]

(c) Let \( B \) be the event that at least one value is observed more than once. We see that \( \mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) \). So we need to find out the probability that the values of the 5 rolls are distinct. We
see that \( P(B) \) simply the number of ways to choose 5 numbers (order matters) divided by the sample space (which is 6^5). So

\[
P(B) = \frac{6!}{6^5} = \frac{5!}{6^4}.
\]

So,

\[
P(B) = 1 - \frac{5!}{6^4}.
\]

5 Past Probabilified

In this question we review some of the past CS70 topics, and look at them probabilistically.

For the following experiments,

i. Define an appropriate sample space \( \Omega \).

ii. Give the probability function \( P(\omega) \).

iii. Compute \( P(E_1) \) given event \( E_1 \).

iv. Compute \( P(E_2) \) given event \( E_2 \).

(a) Fix a prime \( p > 2 \), and uniformly sample twice with replacement from \( \{0, \ldots, p - 1\} \) (assume we have two \( \{0, \ldots, p - 1\} \)-sided fair dice and we roll them). Then multiply these two numbers with each other in \( (\mod p) \) space.

\( E_1 = \) The resulting product is 0.

\( E_2 = \) The product is \( (p - 1)/2 \).

(b) Make a graph on \( n \) vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).

\( E_1 = \) The graph is complete.

\( E_2 = \) vertex \( v_1 \) has degree \( d \).

(c) Use the Stirling’s approximation to simplify \( P(E_2) \) from part b.

(d) Create a random stable matching instance by having each person’s preference list be a random permutation of the opposite entity’s list (make the preference list for each individual job and each individual candidate a random permutation of the opposite entity’s list). Finally, create a uniformly random pairing by matching jobs and candidates up uniformly at random (note that in this pairing, (1) a candidate cannot be matched with two different jobs, and a job cannot be matched with two different candidates (2) the pairing does not have to be stable). \( E_1 = \) All jobs have distinct favorite candidates.

\( E_2 = \) The resulting pairing is the candidate-optimal stable pairing.
(e) Use the Stirling’s approximation to simplify $\mathbb{P}(E_1)$ from part d.

Solution:

(a)  
  i. This is essentially the same as throwing two $\{0, \ldots, p-1\}$-sided dice, so one appropriate sample space is $\Omega = \{(i, j) : i, j \in \text{GF}(p)\}$.

  ii. Since there are exactly $p^2$ such pairs, the probability of sampling each one is $\mathbb{P}[(i, j)] = 1/p^2$.

  iii. Now in order for the product $i \cdot j$ to be zero, at least one of them has to be zero. There are exactly $2p - 1$ such pairs, and so $\mathbb{P}(E_1) = \frac{2p-1}{p^2}$.

  iv. For $i \cdot j$ to equal $(p-1)/2$ it doesn’t matter what $i$ is as long as $i \neq 0$ and $j \equiv i^{-1}(p-1)/2 \pmod{p}$. Thus $|E_2| = |\{(i, j) : j \equiv i^{-1}(p-1)/2\}| = p - 1$, and whence $\mathbb{P}(E_2) = \frac{p-1}{p^2}$.

Alternative Solution for $\mathbb{P}(E_2)$: The previous reasoning showed that $(p-1)/2$ is in no way special, and the probability that $i \cdot j = (p-1)/2$ is the same as $\mathbb{P}(i \cdot j = k)$ for any $k \in \text{GF}(p)$. But $1 = \sum_{k=0}^{p-1} \mathbb{P}(i \cdot j = k) = \mathbb{P}(i \cdot j = 0) + (p-1)\mathbb{P}(i \cdot j = (p-1)/2) = \frac{2p-1}{p^2} + (p-1)\mathbb{P}(i \cdot j = (p-1)/2)$, and so $\mathbb{P}(E_2) = \left(1 - \frac{2p-1}{p^2}\right)/(p-1) = \frac{n-1}{p^2}$ as desired.

(b) i. Since any $n$-vertex graph can be sampled, $\Omega$ is the set of all graphs on $n$ vertices.

  ii. As there are $N = 2^n$ such graphs, the probability of each individual one $g$ is $\mathbb{P}(g) = 1/N$ (by the same reasoning that every sequence of fair coin flips is equally likely!).

  iii. There is only one complete graph on $n$ vertices, and so $\mathbb{P}(E_1) = 1/N$.

iv. For vertex $v_1$ to have degree $d$, exactly $d$ of its $n - 1$ possible adjacent edges must be present. There are $\binom{n-1}{d}$ choices for such edges, and for any fixed choice, there are $2^n - (n-1)$ graphs with this choice. So $\mathbb{P}(E_2) = \frac{\binom{n-1}{d}2^n - (n-1)}{2^n} = \binom{n-1}{d} \left(\frac{1}{2}\right)^{n-1}$.

(c) Since $\mathbb{P}(E_2) = \binom{n-1}{d} \left(\frac{1}{2}\right)^{n-1} = \frac{(n-1)!}{d!(n-1-d)!} \left(\frac{1}{2}\right)^{n-1}$. From the Stirling’s Approximation we get

$$\mathbb{P}(E_2) = \frac{(n-1)!}{d!(n-1-d)!} \left(\frac{1}{2}\right)^{n-1} \approx \left(\frac{n-1}{e}\right)^{n-1} \left(\frac{e}{n-d-1}\right)^{n-d-1} \left(\frac{e}{d}\right)^d \left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{n-1}{e}\right)^{n-1} \left(\frac{1}{n-d-1}\right)^{n-1} \left(\frac{d}{e}\right)^d \left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{n-1}{n-d-1}\right)^{n-d-1} \left(\frac{d}{e}\right)^d \left(\frac{1}{2}\right)^{n-1}.$$

Consider $m = \frac{d}{n-1} \leq 1$. So we can simplify the expression to

$$\mathbb{P}(E_2) \approx \left(\frac{n-1}{n-d-1}\right)^{n-d-1} \left(\frac{d}{e}\right)^d \left(\frac{1}{2}\right)^{n-1} \approx \left(\frac{1-m}{1-m}\right)^{m(n-1)} \left(\frac{1}{m}\right)^{m(n-1)} \left(\frac{1}{2}\right)^{n-1}.$$

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which yields to

\[ \mathbb{P}(E_2) \approx \left( \frac{1}{1-m} \right)^{1-m} \left( \frac{1}{m} \right)^m \left( \frac{1}{2} \right)^{n-1} \]

Since \(0 \leq m \leq 1\) the first two terms in the brackets are \(0 \leq \left( \frac{1}{1-m} \right)^{1-m} \left( \frac{1}{m} \right)^m < 2\). Thus,

\[ 0 \leq \left( \frac{1}{1-m} \right)^{1-m} \left( \frac{1}{m} \right)^m \left( \frac{1}{2} \right) < 1. \]

This means the probability of \(E_2\) exponentially decreases with increase of \(n\).

\[ \lim_{n \to \infty} \mathbb{P}(E_2) \approx \left( \frac{1}{1-m} \right)^{1-m} \left( \frac{1}{m} \right)^m \left( \frac{1}{2} \right)^{n-1} = 0 \]

(d) i. Here there are two random things we need to keep track of: The random preference lists and the random pairing. A person \(i\)'s preference list can be represented as a permutation \(\sigma_i\) of \(\{1, \ldots, n\}\), and the pairing itself is encoded in another permutation \(\rho\) of the same set (indicating that job \(i\) is paired with candidate \(\rho(i)\)). So \(\Omega = \{(\sigma_1, \ldots, \sigma_{2n}, \rho) : \sigma_i, \rho \in S_n\}\), where \(S_n\) is the set of permutations of \(\{1, \ldots, n\}\).

ii. \(|\Omega| = (n!)^{2n+1}\), and so \(\mathbb{P}(\mathcal{P}) = 1/|\Omega|\) for each \(\mathcal{P} \in \Omega\).

iii. For \(E_1\), we observe that there are \(n!\) possible configurations of all jobs having distinct favourite candidates, and that each job has \((n-1)!\) ways of ordering their non-favourite candidates, so \(|E_1| = \frac{n!}{\rho} \cdot \left(\frac{(n-1)!}{n!}\right)^n \cdot \left(\frac{n^n}{n!}\right)\). Consequently, \(\mathbb{P}(E_1) = n! \left(\frac{(n-1)!}{n!}\right)^n = \frac{n!}{n^{2n}}\).

iv. No matter what \(\sigma_1, \ldots, \sigma_{2n}\) are, there is exactly one candidate-optimal pairing, and so \(\mathbb{P}(E_2) = \frac{(n!)^{2n}}{(n!)^{2n+1}} = \frac{1}{n!}\).

(e) We have \(\mathbb{P}(E_1) = \frac{n!}{n^n}\). From the Srling’s Approximation we get

\[ \mathbb{P}(E_1) = \frac{n!}{n^n} \approx \left(\frac{n}{e}\right)^n = \left(\frac{1}{e}\right)^n \]  

This means the probability of \(E_1\) exponentially decreases with increase of \(n\).

6 Peaceful rooks

A friend of yours, Eithen Quinn, is fascinated by the following problem: placing \(m\) rooks on an \(n \times n\) chessboard, so that they are in peaceful harmony (i.e. no two threaten each other). Each
rook is a chess piece, and two rooks threaten each other if and only if they are in the same row or column. You remind your friend that this is so simple that a baby can accomplish the task. You forget however that babies cannot understand instructions, so when you give the $m$ rooks to your baby niece, she simply puts them on random places on the chessboard. She however, never puts two rooks at the same place on the board.

(a) Assuming your niece picks the places uniformly at random, what is the chance that she places the $(i+1)^{\text{st}}$ rook such that it doesn’t threaten any of the first $i$ rooks, given that the first $i$ rooks don’t threaten each other?

(b) What is the chance that your niece actually accomplishes the task and does not prove you wrong?

(c) Now imagine that the rooks can be stacked on top of each other, then what would be the probability that your niece’s placements result in peace? Assume that two rooks threaten each other if they are in the same row or column. Also two pieces stacked on top of each other are obviously in the same row and column, therefore they threaten each other.

(d) Explain the relationship between your answer to the previous part and the birthday paradox. In particular if we assume that 23 people have a 50% chance of having a repeated birthday (in a 365-day calendar), what is the probability that your niece places 23 stackable pieces in a peaceful position on a $365 \times 365$ board?

Solution:

(a) After having placed $i$ rooks in a peaceful position, $i$ of the rows and $i$ of the columns are taken. So for the next rook we have $n-i$ choices for the row and $n-i$ choices for the column in order to remain in a peaceful position. The total number of board cells left is $n^2-i$. So the chance that the next rook keeps the peace is $\frac{(n-i)^2}{n^2-i}$.

(b) The product over $i=0,\ldots,m-1$ gives us the final answer. So the answer is

$$\prod_{i=0}^{m-1} \frac{(n-i)^2}{n^2-i} = \frac{(n!)^2(n^2-m)!}{(n^2)!((n-m)!)^2}$$

(c) The only thing that changes from the previous part is that when placing the $i$-th piece, we no longer have $n^2-i$ possibilities, but $n^2$ possibilities. So the answer changes to

$$\prod_{i=0}^{m-1} \frac{(n-i)^2}{n^2} = \frac{(n!)^2}{((n-m)!)^2n^{2m}}$$

(d) The columns must be different.

All the rows being different is simply the birthday paradox. Similarly all the columns being different is another birthday paradox. So if the probability that $m$ persons have different birthdays in an $n$-day calendar is $p$, then the probability that $m$ rooks end up in a peaceful position
on an $n \times n$ chessboard is $p^2$. Of course this can be verified by hand. The answer to the previous part is

$$\frac{(n!)^2}{((n-m)!)^2 n^2m} = \left(\frac{n!}{(n-m)!n^m}\right)^2$$

The expression inside the parenthesis is the answer to the birthday paradox.

So if the probability $p$ is 0.5 (which roughly happens for $n = 365$ and $m = 23$), then the probability that rooks end up in a peaceful position is $p^2 = 0.25$.

7 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world’s Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly’s factory was poisonous. At See’s factory, 1.5% of Smarties produced were poisonous. And at Furd’s factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn’t come from Burr Kelly’s factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd’s Smarties Factory?

Solution:

(a) Let $S$ be the event that a smarty is safe to eat.

Let $BK$ be the event that a smarty is from Burr Kelly’s factory.

Let $YS$ be the event that a smarty is from Yousef See’s factory.

Finally, let $SF$ be the event that a smarty is from Stan Furd’s factory.

$$\mathbb{P}(S) = \mathbb{P}(BK)\mathbb{P}(S | BK) + \mathbb{P}(YS)\mathbb{P}(S | YS) + \mathbb{P}(SF)\mathbb{P}(S | SF)$$

$$= (0.45)(0.99) + (0.35)(0.985) + (0.2)(0.98) = 0.98625.$$

Therefore the probability that a Smarty is safe to eat is about 0.98625.
(b) Let $P$ be the event that a smarty is poisonous.

$$
\mathbb{P}(P \mid \neg BK) = \mathbb{P}(YS \mid \neg BK)\mathbb{P}(P \mid YS) + \mathbb{P}(SF \mid \neg BK)\mathbb{P}(P \mid SF)
$$

$$
= \frac{\mathbb{P}(YS)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid YS) + \frac{\mathbb{P}(SF)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid SF)
$$

$$
= \frac{0.35}{0.55} \cdot 0.015 + \frac{0.2}{0.55} \cdot 0.02 = 0.0168.
$$

(c)

$$
\mathbb{P}(SF \mid P) = \frac{\mathbb{P}(P \mid SF)\mathbb{P}(SF)}{\mathbb{P}(P)}
$$

In the first part we calculate the probability that any random Smarty was safe to eat. We can use that since $\mathbb{P}(P) = 1 - \mathbb{P}(S)$. Therefore the solution becomes:

$$
\mathbb{P}(SF \mid P) = \frac{\mathbb{P}(P \mid SF)\mathbb{P}(SF)}{1 - \mathbb{P}(S)}
$$

$$
= \frac{(0.02)(0.2)}{1 - 0.98625} = 0.29.
$$

8 Make Your Own Question

You must make your own question on this week’s material and solve it.

9 Homework Process and Study Group

You must describe your homework process and study group in order to receive credit for this question.