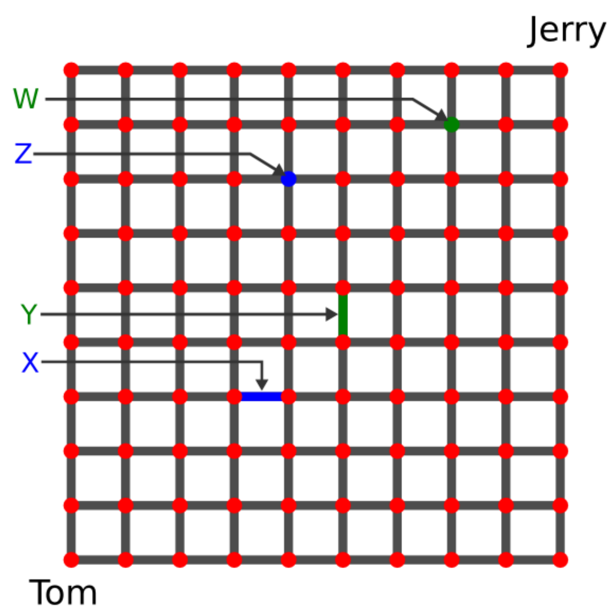
Due: Fri, 3/20 11:59 PM
Grace period until Sun 3/22 11:59 PM

1 Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.



(a) How many such shortest paths exist?

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(c) How many shortest paths pass through the vertex labeled Z? The vertex labeled W? Both the vertices Z and W? Neither vertex Z nor vertex W?

2 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\binom{n}{i} = \binom{n}{n-i}$.

(b) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$. (Hint: Part (a) might be useful.)

(c) $\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n-i}\binom{n-i}{j} = 3^n$. (Hint: consider the number of ways of splitting n elements into 3 groups.)

3 Fibonacci Fashion

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1,2,\ldots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\ldots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \ge 0}\sum_{x_2 \ge 0}\cdots\sum_{x_t \ge 0}\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

4 Probability Warm-Up

(a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls out of the bucket, what is the probability of getting exactly k red balls (assuming $0 \le k \le 20$) if the sampling is done **with** replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?

(b) Same as part (a), but the sampling is **without** replacement, i.e. after we take a ball out the bucket we **do not** return the ball back to the bucket.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

5 Past Probabilified

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}(\omega)$.

iii. Compute $\mathbb{P}(E_1)$ given event E_1.

iv. Compute $\mathbb{P}(E_2)$ given event E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0, \dots, p-1\}$ (assume we have two $\{0, \dots, p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $(\bmod p)$ space.
$E_1 =$ The resulting product is 0.
$E_2 =$ The product is $(p-1)/2$.

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).
$E_1 =$ The graph is complete.
$E_2 =$ vertex v_1 has degree d.

(c) Use the Stirling's approximation to simplify $\mathbb{P}(E_2)$ from part b.

(d) Create a random stable matching instance by having each person's preference list be a random permutation of the opposite entity's list (make the preference list for each individual job and each individual candidate a random permutation of the opposite entity's list). Finally, create a uniformly random pairing by matching jobs and candidates up uniformly at random (note that in this pairing, (1) a candidate cannot be matched with two different jobs, and a job cannot be matched with two different candidates (2) the pairing does not have to be stable). $E_1 =$ All jobs have distinct favorite candidates.
$E_2 =$ The resulting pairing is the candidate-optimal stable pairing.

(e) Use the Stirling's approximation to simplify $\mathbb{P}(E_1)$ from part d.

6 Peaceful rooks

A friend of yours, Eithen Quinn, is fascinated by the following problem: placing m rooks on an $n \times n$ chessboard, so that they are in peaceful harmony (i.e. no two threaten each other). Each rook is a chess piece, and two rooks threaten each other if and only if they are in the same row or column. You remind your friend that this is so simple that a baby can accomplish the task. You forget however that babies cannot understand instructions, so when you give the m rooks to your baby niece, she simply puts them on random places on the chessboard. She however, never puts two rooks at the same place on the board.

(a) Assuming your niece picks the places uniformly at random, what is the chance that she places the $(i+1)^{\text{st}}$ rook such that it doesn't threaten any of the first i rooks, given that the first i rooks don't threaten each other?

(b) What is the chance that your niece actually accomplishes the task and does not prove you wrong?

(c) Now imagine that the rooks can be stacked on top of each other, then what would be the probability that your niece's placements result in peace? Assume that two rooks threaten each other if they are in the same row or column. Also two pieces stacked on top of each other are obviously in the same row and column, therefore they threaten each other.

(d) Explain the relationship between your answer to the previous part and the birthday paradox. In particular if we assume that 23 people have a 50% chance of having a repeated birthday (in a 365-day calendar), what is the probability that your niece places 23 stackable pieces in a peaceful position on a 365×365 board?

7 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

8 Make Your Own Question

Make your own question on this week's material and solve it.

9 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

1. **What sources (if any) did you use as you worked through the homework?**

2. **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
3. **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
4. **Roughly how many total hours did you work on this homework?**