Note 8 Supplement: Polynomial Division Theorem

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The purpose of this note is to give a proof of the following:

Theorem 1. *Let A and B be polynomials over a field, where B is not the zero polynomial. Then there exist unique polynomials Q and R such that $A = QB + R$, where* $\deg R < \deg B$.

Proof. Let $d_1 := \deg A$ and $d_2 := \deg B$. We will prove the statement by induction on the variable d_1. First, we dispense of a special case. If A is the zero polynomial, then we can take $Q = R = 0$. Next, the base case is $d_1 = 0$. If $d_1 = 0$, then $d_2 = 0$ as well, i.e., both A and B are constants, but then we can take $Q = A/B$ and $R = 0$.

So, now fix a positive integer d_1 and suppose that whenever A' and B' are polynomials of degrees d_1' and d_2' respectively, with $d_1' < d_1$, then there are polynomials Q' and R' such that $A' = Q'B' + R'$ with $\deg R' < \deg B'$ (notice that we are using a strong inductive hypothesis).

Case 1: $d_1 < d_2$. If so, we can take $Q = 0$ and $R = A$.

Case 2: $d_1 \ge d_2$. We can write

$$A(x) = a_{d_1}x^{d_1} + \cdots + a_1 x + a_0,$$
$$B(x) = b_{d_2}x^{d_2} + \cdots + b_1 x + b_0,$$

for coefficients $a_0, a_1, \ldots, a_{d_1}, b_0, b_1, \ldots, b_{d_2}$, where $a_{d_1} \neq 0$ and $b_{d_2} \neq 0$. Now define the polynomial $A'(x) := A(x) - (a_{d_1}/b_{d_2})x^{d_1-d_2}B(x)$. Notice that $(a_{d_1}/b_{d_2})x^{d_1-d_2}B(x)$ has the same leading term as $A(x)$, so the subtraction of this term kills off the leading term of A, leaving $\deg A' < \deg A$. By the strong inductive hypothesis, there exist polynomials Q' and R, with $\deg R < \deg B$, such that $A' = Q'B + R$. Therefore,

$$A(x) = \underbrace{\left(\frac{a_{d_1}}{b_{d_2}} x^{d_1 - d_2} + Q'(x)\right)}_{Q(x)} B(x) + R(x)$$

which proves the existence of the polynomials Q and R.

To prove uniqueness, suppose that $A = Q_1 B + R_1 = Q_2 B + R_2$ for polynomials Q_1, Q_2, R_1, R_2, where $\deg R_1 < B$ and $\deg R_2 < B$. Then, we have $(Q_1 - Q_2)B = R_2 - R_1$. Now we compute the degree of each side. The degree of the right hand side is strictly less than $\deg B$, but if $Q_1 \neq Q_2$, then the left hand side has degree which is at least $\deg B$ which is impossible. Therefore we must have $Q_1 = Q_2$ and so $R_1 = R_2$. $\square$