Note on binomial random variable

Ref: Ross, A First Course in Probability, pp. 134–135, 8th Edition

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The probability mass function of a binomial random variable $X \sim \text{Binomial}(n,p)$ is:

$$\Pr[X=i] = \binom{n}{i} p^i (1-p)^{n-i} \qquad i = 0, 1, \ldots, n.$$

Why?

- The probability of any sequence of n outcomes with i successes and $n-i$ failures is $p^i (1-p)^{n-i}$ because the trials are assumed to be independent.

e.g. Suppose you flip a biased coin with $\Pr[\text{heads}] = p$ and $\Pr[\text{tails}] = 1-p$ three times. Compute the probability of any particular sequence with 2 heads and 1 tail.

$$\left.\begin{aligned} \Pr[HHT] &= p^2(1-p) \\ \Pr[HTH] &= p(1-p)p \\ \Pr[THH] &= (1-p)p^2 \end{aligned}\right\} \text{ each has probability } = p^2(1-p). (*)$$

in further detail

$$\begin{aligned} \Pr[HHT] &= \Pr[\text{heads on } 1^{st} \text{ flip} \cap \text{heads on } 2^{nd} \text{ flip} \cap \text{tails on } 3^{rd} \text{ flip}] \\ &= \Pr[\text{heads on } 1^{st} \text{ flip}] \Pr[\text{heads on } 2^{nd} \text{ flip}] \Pr[\text{tails on } 3^{rd} \text{ flip}] \\ &= p \cdot p \cdot (1-p). \end{aligned}$$

outcome of one flip does not influence the outcome of another

- There are $\binom{n}{i}$ distinct sequences of the n outcomes leading to i successes and $n-i$ failures.

e.g. Continuing the example above, $X \sim \text{Binomial}(3, p)$

3 flips

probability of heads on any flip is p

$$\Pr[X=2] = \Pr[HHT \cup HTH \cup THH] = \Pr[HHT] + \Pr[HTH] + \Pr[THH]$$

getting 2 heads

3 different ways to get 2 heads

each way is distinct

$$\begin{aligned} &= p^2(1-p) + p^2(1-p) + p^2(1-p) \quad \text{see } * \\ &= 3p^2(1-p) \\ &= \binom{3}{2} p^2(1-p). \end{aligned}$$

$3 = \binom{3}{2}$

Prepared by: M. Chapman 7/16/2017